Small Scale Self-Alignment with the SVT

G. Van Buren†, Y. Fisyak†, S. Margetis‡, V. Perevoztchikov†
†Brookhaven National Laboratory, Upton, New York 11973
‡Kent State University, Kent, Ohio 44242

February 24, 2006

Abstract

This document describes how to self-align each wafer of the STAR SVT at small scales using 6 quantities obtained by fitting tracks to only the SVT points and a primary vertex: the 2 components of the local wafer coordinates of the hit from the track (A, B), the residuals from the fit to the hit position $(\Delta A_{obs}, \Delta B_{obs})$, and 2 quantities which describe the directionality of the fit track through the wafer. Relative calibration of the drift velocities of each hybrid on a wafer may also be extracted. Two additional quantities representing the resolutions of measuring A and B may also be necessary.

1 Introduction

In pursuit of calibrating the alignment of SVT wafers to the maximal precision, it is best to exclude the use of any other detectors which may bias or degrade the calibration. To this end, it appears that a self-alignment procedure using only hits in the SVT is optimal. This can be done using residuals of hits made by particle tracks passing through the wafers. Doing so with helical tracks (in a magnetic field) means that it is necessary to have at least 4 points on a track to prevent over-constraint of the helical fit (3 points defines a circle in the transverse plane, leaving no residuals in that plane). This can be accomplished using hits in all 3 layers of the SVT along with a precise primary vertex point.

The proposed method is to use tracks from the STAR TPC to locate SVT hits which belong to primary tracks, along with a primary vertex. Where 3 SVT hits are found, those hits and the primary vertex are then re-fit to a helix, and the residuals to SVT hits are used in the technique which we will describe here. We will utilize linear approximations to several dependencies between observable quantities and the related alignment parameters, so it is a requirement that any existing misalignments must be very small. If they are not, iterating the method may be necessary. Use of the same technique in the CMS experiment shows that convergence is mostly achieved after a single iteration, but that 3-4 iterations complete the calibration [1].

It should be explicitly stated at this point that this technique has only one weak constraint to prevent it from altering the global rotation of the SVT: the use of multiple events with a significant spread in primary vertex locations which serve as fixed references to the global coordinate system (this is not sufficiently true in the transverse plane). Otherwise, the χ^2 minimization used in this technique has the potential to rotate the entire system. It is therefore recommended that a global alignment procedure be done before this self-alignment procedure, and checked again afterwards. It is also necessary to use a selection of many events with varied primary vertex locations (at least in longitude), and that the tracks used in calibrating each wafer cross a variety of other wafers (to avoid calibrating a subset of wafers without constraint to the full SVT).

Performing this calibration in STAR with straight tracks in zero magnetic field has been previously proposed [2]. However, using helical tracks with the magnetic field on is beneficial in allowing a selection of tracks with high enough momentum to discard those which might have large multiple scattering contributions. Curved tracks also provide for a slightly greater distribution of track trajectories through wafers, increasing the number of possible wafer combinations and lengthening the lever arm of incident angles to wafers (which we will see later is necessary for calibrations normal to the surface of the wafers), although the highest curvature tracks (lowest transverse momentum) are the ones removed for multiple scattering minimization.

It is also worthwhile to note that we see nothing to prevent inclusion of the STAR SSD in this procedure.

2 Coordinates

In STAR global coordinates, an SVT hit is defined as:

$$\vec{h} \quad = \quad \vec{x} + A\hat{d} + B\hat{t} \tag{1}$$

The vector $\vec{x}$ is the global position of the center of the wafer, $\hat{d}$ is the drift direction within the wafer, and $\hat{t}$ is the the direction transverse to the drift but in the plane of the wafer. The SVT group also defines $\hat{n}$ which is the normal vector to the surface of the wafer. Here, $\hat{t} = \hat{d} \times \hat{n}$. A and B are the SVT measures of the hit location in the plane of the wafer (the $\hat{d}\hat{t}$ plane).

We define the global coordinates in which $\vec{x}, \hat{d}, \hat{n}, \hat{t}$ are stored in the database to be $\hat{i}, \hat{j}, \hat{k}$ to avoid confusion over the definition of x. But we choose to work in the wafer coordinate space $\hat{d}, \hat{n}, \hat{t}$ for simplicity in our solution. To convert $\vec{x}$

back and forth between the global and wafer coordinate spaces, we have

$$\hat{d} = d_i\hat{i} + d_j\hat{j} + d_k\hat{k} \tag{2}$$

$$\hat{n} = n_i\hat{i} + n_j\hat{j} + n_k\hat{k} \tag{3}$$

$$\hat{t} = t_i\hat{i} + t_j\hat{j} + t_k\hat{k} \tag{4}$$

$$\vec{x} = x_i\hat{i} + x_j\hat{j} + x_k\hat{k} \tag{5}$$

$$= x_d\hat{d} + x_n\hat{n} + x_t\hat{t} \tag{6}$$

$$= (\vec{x}\cdot\hat{d})\hat{d} + (\vec{x}\cdot\hat{n})\hat{n} + (\vec{x}\cdot\hat{t})\hat{t} \tag{7}$$

$$= \begin{array}{l} (x_i * d_i + x_j * d_j + x_k * d_k)\hat{d} \\ +\, (x_i * n_i + x_j * n_j + x_k * n_k)\hat{n} \\ +\, (x_i * t_i + x_j * t_j + x_k * t_k)\hat{t} \end{array} \tag{8}$$

One should understand that $\vec{x}$ is a vector from the origin of the global coordinate space to the origin of the wafer coordinate space, so it is not necessary for any of $x_d, x_n, x_t, x_i, x_j, x_k$ to be zero, and the magnitude $|\vec{x}|$ should be the same in either coordinate space. Using $i_d = (\hat{i}\cdot\hat{d}) = (\hat{d}\cdot\hat{i}) = d_i$, etc., we can also write

$$\vec{x} = (\vec{x}\cdot\hat{i})\hat{i} + (\vec{x}\cdot\hat{j})\hat{j} + (\vec{x}\cdot\hat{k})\hat{k} \tag{9}$$

$$= \begin{array}{l} (x_d * i_d + x_n * i_n + x_t * i_t)\hat{d} \\ +\, (x_d * j_d + x_n * j_n + x_t * j_t)\hat{n} \\ +\, (x_d * k_d + x_n * k_n + x_t * k_t)\hat{t} \end{array} \tag{10}$$

$$= \begin{array}{l} (x_d * d_i + x_n * n_i + x_t * t_i)\hat{d} \\ +\, (x_d * d_j + x_n * n_j + x_t * t_j)\hat{n} \\ +\, (x_d * d_k + x_n * n_k + x_t * t_k)\hat{t} \end{array} \tag{11}$$

3 Small shifts

3.1 Rotations

We can envision rotations of the wafer on each of the three axes $\hat{d}, \hat{n}, \hat{t}$ about the center of the wafer. The associated small rotation angles we shall call $\delta\phi_d, \delta\phi_n, \delta\phi_t$ and we can treat them independently to first order.

For each small rotation of the plane, the vector to the hit from the center of the plane ($\vec{h} - \vec{x} = A\hat{d} + B\hat{t}$) is rotated by a matrix:

$$R_{\delta\phi} = \begin{pmatrix} \cos(\delta\phi) & \sin(\delta\phi) \\ -\sin(\delta\phi) & \cos(\delta\phi) \end{pmatrix} \approx \begin{pmatrix} 1 & \delta\phi \\ -\delta\phi & 1 \end{pmatrix} \tag{12}$$

For each of the small rotations, we apply the rotation to the projection of $A\hat{d}+B\hat{t}$ which lies in the plane being rotated. So, for the rotation of $\Delta\vec{h}_{\delta\phi_d}$ about the $\hat{d}$

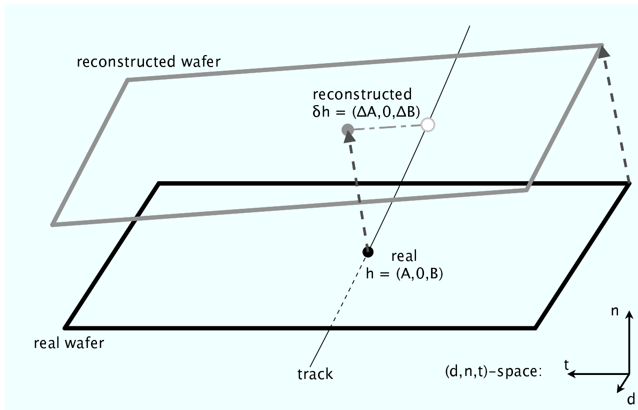

Figure 1: Example of real and reconstructed hit positions and track propagation in the wafer coordinate system. Dashed arrows represent the small shift transformation.

axis, we use the projection to the $\hat{n}\hat{t}$ plane, etc., as follows:

$$I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Delta\vec{h}_{\delta\phi_d} = (R_{\delta\phi_d} - I)\begin{pmatrix} 0 \\ B \end{pmatrix} \begin{matrix} \hat{n} \\ \hat{t} \end{matrix} = \begin{pmatrix} B\delta\phi_d \\ 0 \end{pmatrix} \begin{matrix} \hat{n} \\ \hat{t} \end{matrix} \tag{13}$$

$$\Delta\vec{h}_{\delta\phi_n} = (R_{\delta\phi_n} - I)\begin{pmatrix} B \\ A \end{pmatrix} \begin{matrix} \hat{t} \\ \hat{d} \end{matrix} = \begin{pmatrix} A\delta\phi_n \\ -B\delta\phi_n \end{pmatrix} \begin{matrix} \hat{t} \\ \hat{d} \end{matrix} \tag{14}$$

$$\Delta\vec{h}_{\delta\phi_t} = (R_{\delta\phi_t} - I)\begin{pmatrix} A \\ 0 \end{pmatrix} \begin{matrix} \hat{d} \\ \hat{n} \end{matrix} = \begin{pmatrix} 0 \\ -A\delta\phi_t \end{pmatrix} \begin{matrix} \hat{d} \\ \hat{n} \end{matrix} \tag{15}$$

Some of these cause out-of-plane shifts with respect to the wafer, and we will handle this later in Section 3.3.

The result of all rotations is then the linear sum:

$$\Delta\vec{h}_{\text{rotations}} = \Delta\vec{h}_{\delta\phi_d} + \Delta\vec{h}_{\delta\phi_n} + \Delta\vec{h}_{\delta\phi_t} \tag{16}$$

$$= (-B\delta\phi_n)\hat{d} + (-A\delta\phi_t + B\delta\phi_d)\hat{n} + (A\delta\phi_n)\hat{t} \tag{17}$$

For completeness, in this linear approximation one could have alternately written these three rotation matrices as a single 3x3 matrix as follows:

$$R' \approx \begin{pmatrix} 1 & \delta\phi_t & -\delta\phi_n \\ -\delta\phi_t & 1 & \delta\phi_d \\ \delta\phi_n & -\delta\phi_d & 1 \end{pmatrix}, I' \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

$$\Delta\vec{h}_{\text{rotations}} = (R' - I')\begin{pmatrix} A \\ 0 \\ B \end{pmatrix} \begin{matrix} \hat{d} \\ \hat{n} \\ \hat{t} \end{matrix} = \begin{pmatrix} -B\delta\phi_n \\ -A\delta\phi_t + B\delta\phi_d \\ A\delta\phi_n \end{pmatrix} \begin{matrix} \hat{d} \\ \hat{n} \\ \hat{t} \end{matrix} \tag{19}$$

Such 3x3 matrices are often written with the opposite sign on $\delta\phi_n$, but the notation used here is necessary to be consistent with how it was written in Equation 14.

3.2 Translations

We label the small translation shifts to the wafer plane as $\delta x_d, \delta x_n, \delta x_t$. These yield 1-to-1 corresponding shifts in the hit reconstruction.

$$\Delta\vec{h}_{\text{translations}} = (-\delta x_d)\hat{d} + (-\delta x_n)\hat{n} + (-\delta x_t)\hat{t} \tag{20}$$

Summing the rotations and translations gives:

$$\Delta\vec{h} = \Delta\vec{h}_{\text{rotations}} + \Delta\vec{h}_{\text{translations}} \tag{21}$$

$$= (-B\delta\phi_n - \delta x_d)\hat{d} + (-A\delta\phi_t + B\delta\phi_d - \delta x_n)\hat{n} + (A\delta\phi_n - \delta x_t)\hat{t} \tag{22}$$

3.3 Projecting to the wafer

Unfortunately, the reconstructed hit must be within the plane of the wafer, so we cannot see shifts in the $\hat{n}$ direction directly. If we assume the particle track to be linear (which should be a valid approximation for the small distances with which we are working here), then any shift S of the wafer in the $\hat{n}$ direction will lead to a shift of the reconstructed hit in the $\hat{d}$ and $\hat{t}$ directions by amounts proportional to the tangents of the angles θ_{dn}, θ_{tn} between the track vector $\vec{v}$ and the normal vector $\hat{n}$ in the $\hat{d}\hat{n}$ and $\hat{t}\hat{n}$ planes. The projection of a shift $\vec{S} = S\hat{n}$ thus reconstructs as:

$$\vec{v} = v_d\hat{d} + v_n\hat{n} + v_t\hat{t} \tag{23}$$

$$v_{dn} = \tan\theta_{dn} = v_d/v_n \tag{24}$$

$$v_{tn} = \tan\theta_{tn} = v_t/v_n \tag{25}$$

$$\vec{S} = S\hat{n} \Longrightarrow (-v_{dn}S)\hat{d} + (-v_{tn}S)\hat{t} \tag{26}$$

where v_d, v_n, v_t can be obtained from v_i, v_j, v_k just as in Equation 8. In our case, the shift in the $\hat{n}$ direction, which includes both that due to translations *and* rotations is shown in Equation 22 as the $\hat{n}$ component of the full shift $\Delta\vec{h}$ ($S = -A\delta\phi_t + B\delta\phi_d - \delta x_n$). Note that tracks which are normal to the wafers ($v_{dn}, v_{tn} \approx 0$) offer little resolving power for shifts in the $\hat{n}$ direction (and therefore to $\delta\phi_t$, $\delta\phi_d$, and δx_n shifts), which is one of the reasons we benefit from using curved tracks and a wide distribution of primary vertices. After projecting, the reconstructed hit is then actually seen as:

$$\Delta\vec{h}_{reco} = \begin{array}{l}(-B\delta\phi_n - \delta x_d + v_{dn}(A\delta\phi_t - B\delta\phi_d + \delta x_n))\hat{d} \\ + (A\delta\phi_n - \delta x_t + v_{tn}(A\delta\phi_t - B\delta\phi_d + \delta x_n))\hat{t}\end{array} \tag{27}$$

$$= (\Delta A)\hat{d} + (\Delta B)\hat{t} \tag{28}$$

where we use

$$\Delta A = -B\delta\phi_n - \delta x_d + v_{dn}[A\delta\phi_t - B\delta\phi_d + \delta x_n] \tag{29}$$

$$\Delta B = A\delta\phi_n - \delta x_t + v_{tn}[A\delta\phi_t - B\delta\phi_d + \delta x_n] \tag{30}$$

A more formal and rigorous derivation of the dependence of observables on shift quantities is provided in Appendix A.

3.4 Drift Velocities

Additional shifting of hits can result from incorrect drift velocities. These are independent for the two hybrids on a wafer, and are thus two independent quantities. We should more technically write:

$$A = -(\mathcal{A}_{\max} - t\nu_1)\delta_{\text{hyb},1} + (\mathcal{A}_{\max} - t\nu_2)\delta_{\text{hyb},2} \tag{31}$$

using the two drift velocities ν_1 and ν_2 for hybrids 1 and 2 respectively (where hybrid 1 corresponds to $A < 0$), t is the hybrid time bin in which the hit was measured, $\mathcal{A}$ is the maximum drift length (understood to be $\mathcal{A}_{\max} = 3.0\text{cm}$), and $\delta_{\text{hyb},1}$ and $\delta_{\text{hyb},2}$ are Kronecker deltas for the hit to be on hybrid 1 or 2 . We shall call the deviation in drift velocities $\delta\nu_1$ and $\delta\nu_2$, and define $\delta\mu_1, \delta\mu_2$:

$$\begin{aligned} \Delta\vec{h}_{\text{velocities}} &= [-(\delta\nu_1/\nu_1)(\mathcal{A}_{\max} + A)\delta_{\text{hyb},1} + (\delta\nu_2/\nu_2)(\mathcal{A}_{\max} - A)\delta_{\text{hyb},2}]\hat{d} \\ &\equiv [-\delta\mu_1(\mathcal{A}_{\max} + A)\delta_{\text{hyb},1} + \delta\mu_2(\mathcal{A}_{\max} - A)\delta_{\text{hyb},2}]\hat{d} \end{aligned} \tag{32}$$

In reality, deviations in the drift velocities will also distort the observations of some of the other shifts. But we have chosen to work only with first order distortions for this method, so higher order terms will be neglected for now in the expectation that iteration will achieve the desired accuracy.

Our reconstructed hit is now described with Equations 28, 30, and:

$$\begin{aligned} \Delta A &= -B\delta\phi_n - \delta x_d + v_{dn}[A\delta\phi_t - B\delta\phi_d + \delta x_n] \\ &\quad -\delta\mu_1(\mathcal{A}_{\max} + A)\delta_{\text{hyb},1} + \delta\mu_2(\mathcal{A}_{\max} - A)\delta_{\text{hyb},2} \end{aligned} \tag{33}$$

4 Solution

To find the solution, we need to fit the observed shifts (residuals) with the optimal parameterized small shifts discussed in Section 3. Let's call the observed shift $\Delta\vec{h}_{obs}$ and the true shift within our parameterized model $\Delta\vec{h}_{model}$. One might consider minimizing the variance $\mathcal{V}$:

$$\mathcal{V} = \sum_l \left(\left| \Delta\vec{h}_{obs} - \Delta\vec{h}_{model} \right|^2 \right) \tag{34}$$

$$= \sum_l \left((\Delta A_{obs} - \Delta A_{model})^2 + (\Delta B_{obs} - \Delta B_{model})^2 \right) \tag{35}$$

$$= \sum_l \left(\Delta A_{o-m}^2 + \Delta B_{o-m}^2 \right) \tag{36}$$

where

$$\Delta A_{o-m} = (\Delta A_{obs} - \Delta A_{model}) \quad , \quad \Delta B_{o-m} = (\Delta B_{obs} - \Delta B_{model}) \tag{37}$$

and we sum over l tracks (from many events, as stipulated in Section 1) to get $\mathcal{V}$, with the measured quantities A, B, ΔA_{obs}, ΔB_{obs}, v_{dn}, and v_{tn} varying for each track. However, this makes the assumption that the measurements in the $\hat{d}$ and $\hat{t}$ directions have the same resolution, and that those resolutions are identical for every track l. This is not necessarily so. Instead of minimizing the variance in absolute distance, it is more appropriate to minimize the variance normalized to the measurement resolution in each direction for each track used, which is equivalent to χ^2. Thus, we will use:

$$\chi^2 = \sum_l \left(\left(\frac{\Delta A_{o-m}^2}{\sigma_d^2} \right) + \left(\frac{\Delta B_{o-m}^2}{\sigma_t^2} \right) \right) \tag{38}$$

and we will need to determine σ_d^2 and σ_t^2 for use in our calculations (unless it is shown that $\sigma_d^2 = \sigma_t^2$ for all tracks, in which case they can be dropped from our formulas). It might at least be possible to assume that σ_d^2 and σ_t^2 are constants; we will not address that here, but the need to determine these on a track-by-track basis is something which should be considered.

To find the best parameters for our model, we must minimize χ^2 with respect to each small shift parameter (minimizing the difference between the observations and the model) by taking the partial derivative with respect to each parameter and setting it equal to zero. This will give us eight equations from each of the eight partial derivatives with respect to the eight unknown quantities.

So for each shift parameter δq, we have:

$$\begin{aligned} \frac{\partial \chi^2}{\partial \delta q} &= \sum_l \left(\left(\frac{2\Delta A_{o-m}}{\sigma_d^2} \frac{\partial \Delta A_{o-m}}{\partial \delta q} \right) + \left(\frac{2\Delta B_{o-m}}{\sigma_t^2} \frac{\partial \Delta B_{o-m}}{\partial \delta q} \right) \right) \qquad (39) \\ &= \sum_l \left(\left(\frac{-2\Delta A_{o-m}}{\sigma_d^2} \frac{\partial \Delta A_{model}}{\partial \delta q} \right) + \left(\frac{-2\Delta B_{o-m}}{\sigma_t^2} \frac{\partial \Delta B_{model}}{\partial \delta q} \right) \right) \qquad (40) \end{aligned}$$

We can drop the constant factors of -2 when setting equal to zero, giving us:

$$\begin{aligned}
\frac{\partial \chi^2}{\partial \delta \phi_d} = 0 &= \textstyle\sum_l \left(\left(\frac{v_{dn} B \Delta A_{o-m}}{\sigma_d^2} \right) + \left(\frac{v_{tn} B \Delta B_{o-m}}{\sigma_t^2} \right) \right) \qquad (41) \\
\frac{\partial \chi^2}{\partial \delta \phi_n} = 0 &= \textstyle\sum_l \left(\left(\frac{-B \Delta A_{o-m}}{\sigma_d^2} \right) + \left(\frac{A \Delta B_{o-m}}{\sigma_t^2} \right) \right) \qquad (42) \\
\frac{\partial \chi^2}{\partial \delta \phi_t} = 0 &= \textstyle\sum_l \left(\left(\frac{v_{dn} A \Delta A_{o-m}}{\sigma_d^2} \right) + \left(\frac{v_{tn} A \Delta B_{o-m}}{\sigma_t^2} \right) \right) \qquad (43) \\
\frac{\partial \chi^2}{\partial \delta x_d} = 0 &= \textstyle\sum_l \left(-\frac{\Delta A_{o-m}}{\sigma_d^2} \right) \qquad (44) \\
\frac{\partial \chi^2}{\partial \delta x_n} = 0 &= \textstyle\sum_l \left(\left(\frac{v_{dn} \Delta A_{o-m}}{\sigma_d^2} \right) + \left(\frac{v_{tn} \Delta B_{o-m}}{\sigma_t^2} \right) \right) \qquad (45) \\
\frac{\partial \chi^2}{\partial \delta x_t} = 0 &= \textstyle\sum_l \left(-\frac{\Delta B_{o-m}}{\sigma_t^2} \right) \qquad (46) \\
\frac{\partial \chi^2}{\partial \delta \mu_1} = 0 &= \textstyle\sum_l \left(-\frac{(\mathcal{A}_{\max} + A) \delta_{\mathrm{hyb},1} \Delta A_{o-m}}{\sigma_d^2} \right) \qquad (47) \\
\frac{\partial \chi^2}{\partial \delta \mu_2} = 0 &= \textstyle\sum_l \left(\frac{(\mathcal{A}_{\max} - A) \delta_{\mathrm{hyb},2} \Delta A_{o-m}}{\sigma_d^2} \right) \qquad (48)
\end{aligned}$$

This linear system of equations can be written in matrix form as shown in Appendix B, where we have multiplied Equation 41 by -1 to obtain a symmetric matrix. Solving this system of equations for the eight unknown quantities is straightforward using any of a number of techniques (matrix inversion, gaussian elimination, etc.). Alternative uses of the matrix are discussed in Appendix D.

5 Final values

Once we have solved for the eight unknowns $(\delta\phi_d, \delta\phi_n, \delta\phi_t, \delta x_d, \delta x_n, \delta x_t, \delta\mu_1, \delta\mu_2)$ for each wafer, we can determine the new values of $\vec{x}', \hat{d}', \hat{n}', \hat{t}'$ for the database. Defining $\Delta\vec{h}_{reco}$ to be the position of the reconstructed hit minus the position of the track (see Appendix C), $\vec{x}$ in wafer coordinates becomes:

$$\vec{x}' = (x_d + \delta x_d)\hat{d} + (x_n + \delta x_n)\hat{n} + (x_t + \delta x_t)\hat{t} \tag{49}$$

To obtain $\vec{x}'$ in global coordinates, for the database, we apply Equation 11. For the rotations, we proceed using:

$$\vec{d'} = \hat{d} - \delta\phi_n\hat{t} + \delta\phi_t\hat{n} \tag{50}$$

$$\vec{n}' = \hat{n} - \delta\phi_t\hat{d} + \delta\phi_d\hat{t} \tag{51}$$

$$\vec{t'} = \hat{t} - \delta\phi_d\hat{n} + \delta\phi_n\hat{d} \tag{52}$$

This math can be done in global coordinates. To first order, this will retain the unitarity of $\vec{d'}, \vec{n}', \vec{t'}$, but if this calibration is done repeatedly, the small deviations from unitarity may become less negligible. Therefore, it is prudent at this point to renormalize by dividing each by their magnitude:

$$\hat{d}' = \vec{d'}/|\vec{d'}| \tag{53}$$

$$\hat{n}' = \vec{n}'/|\vec{n}'| \tag{54}$$

$$\hat{t}' = \vec{t'}/|\vec{t'}| \tag{55}$$

And lastly we modify the drift velocities:

$$\nu_1' = \nu_1(1 + \delta\mu_1) \tag{56}$$

$$\nu_2' = \nu_2(1 + \delta\mu_2) \tag{57}$$

We then enter $\vec{x}', \hat{d}', \hat{n}', \hat{t}'$ for each wafer, and ν_1', ν_2' for each hybrid into the database.

References

[1] V. Karimaki *et al*, proceedings of Vertex 2004 Workshop, not yet published *http://sucimaweb.dipscfm.uninsubria.it/vertex04/upload/14/0/vertex2004_karimaki.pdf*

[2] O. Barranikova *et al*, STAR Note 356

A Formal derivation of shift observables

To be included...

B Solution matrix

$$
\sum_l \left(\begin{matrix}
\left(\frac{v_{dn}^2 B^2}{\sigma_d^2}+\frac{v_{tn}^2 B^2}{\sigma_t^2}\right) & \left(\frac{v_{dn} B^2}{\sigma_d^2}-\frac{v_{tn} AB}{\sigma_t^2}\right) & \left(\frac{-v_{dn}^2 AB}{\sigma_d^2}+\frac{-v_{tn}^2 AB}{\sigma_t^2}\right) & \left(\frac{v_{dn} B}{\sigma_d^2}\right) & \left(\frac{-v_{dn}^2 B}{\sigma_d^2}+\frac{-v_{tn}^2 B}{\sigma_t^2}\right) \\
\left(\frac{v_{dn} B^2}{\sigma_d^2}-\frac{v_{tn} AB}{\sigma_t^2}\right) & \left(\frac{B^2}{\sigma_d^2}+\frac{A^2}{\sigma_t^2}\right) & \left(\frac{-v_{dn} AB}{\sigma_d^2}+\frac{v_{tn} A^2}{\sigma_t^2}\right) & \left(\frac{B}{\sigma_d^2}\right) & \left(\frac{-v_{dn} B}{\sigma_d^2}+\frac{v_{tn} A}{\sigma_t^2}\right) \\
\left(\frac{-v_{dn}^2 AB}{\sigma_d^2}+\frac{-v_{tn}^2 AB}{\sigma_t^2}\right) & \left(\frac{-v_{dn} AB}{\sigma_d^2}+\frac{v_{tn} A^2}{\sigma_t^2}\right) & \left(\frac{v_{dn}^2 A^2}{\sigma_d^2}+\frac{v_{tn}^2 A^2}{\sigma_t^2}\right) & \left(-\frac{v_{dn} A}{\sigma_d^2}\right) & \left(\frac{v_{dn}^2 A}{\sigma_d^2}+\frac{v_{tn}^2 A}{\sigma_t^2}\right) \\
\left(\frac{v_{dn} B}{\sigma_d^2}\right) & \left(\frac{B}{\sigma_d^2}\right) & \left(-\frac{v_{dn} A}{\sigma_d^2}\right) & \left(\frac{1}{\sigma_d^2}\right) & \left(-\frac{v_{dn}}{\sigma_d^2}\right) \\
\left(\frac{-v_{dn}^2 B}{\sigma_d^2}+\frac{-v_{tn}^2 B}{\sigma_t^2}\right) & \left(\frac{-v_{dn} B}{\sigma_d^2}+\frac{v_{tn} A}{\sigma_t^2}\right) & \left(\frac{v_{dn}^2 A}{\sigma_d^2}+\frac{v_{tn}^2 A}{\sigma_t^2}\right) & \left(-\frac{v_{dn}}{\sigma_d^2}\right) & \left(\frac{v_{dn}^2}{\sigma_d^2}+\frac{v_{tn}^2}{\sigma_t^2}\right) \\
\left(\frac{v_{tn} B}{\sigma_t^2}\right) & \left(-\frac{A}{\sigma_t^2}\right) & \left(-\frac{v_{tn} A}{\sigma_t^2}\right) & (0) & \left(-\frac{v_{tn}}{\sigma_t^2}\right) \\
\left(\frac{v_{dn} B(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) & \left(\frac{B(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) & \left(-\frac{v_{dn} A(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) & \left(\frac{(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) & \left(-\frac{v_{dn}(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) \\
\left(-\frac{v_{dn} B(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right) & \left(-\frac{B(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right) & \left(\frac{v_{dn} A(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right) & \left(-\frac{(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right) & \left(\frac{v_{dn}(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right)
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
\left(\frac{v_{tn} B}{\sigma_t^2}\right) & \left(\frac{v_{dn} B(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) & \left(-\frac{v_{dn} B(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right) \\
\left(-\frac{A}{\sigma_t^2}\right) & \left(\frac{B(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) & \left(-\frac{B(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right) \\
\left(-\frac{v_{tn} A}{\sigma_t^2}\right) & \left(-\frac{v_{dn} A(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) & \left(\frac{v_{dn} A(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right) \\
(0) & \left(\frac{(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) & \left(-\frac{(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right) \\
\left(-\frac{v_{tn}}{\sigma_t^2}\right) & \left(-\frac{v_{dn}(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) & \left(\frac{v_{dn}(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right) \\
\left(\frac{1}{\sigma_t^2}\right) & (0) & (0) \\
(0) & \left(\frac{(\mathcal{A}_{\max}+A)^2\delta_{\text{hyb},1}}{\sigma_d^2}\right) & (0) \\
(0) & (0) & \left(\frac{(\mathcal{A}_{\max}-A)^2\delta_{\text{hyb},2}}{\sigma_d^2}\right)
\end{matrix}\right)
\begin{pmatrix} \delta\phi_d \\ \delta\phi_n \\ \delta\phi_t \\ \delta x_d \\ \delta x_n \\ \delta x_t \\ \delta\mu_1 \\ \delta\mu_2 \end{pmatrix}
= \sum_l \begin{pmatrix}
\left(\frac{-v_{dn} B\Delta A_{obs}}{\sigma_d^2}+\frac{-v_{tn} B\Delta B_{obs}}{\sigma_t^2}\right) \\
\left(\frac{-B\Delta A_{obs}}{\sigma_d^2}+\frac{A\Delta B_{obs}}{\sigma_t^2}\right) \\
\left(\frac{v_{dn} A\Delta A_{obs}}{\sigma_d^2}+\frac{v_{tn} A\Delta B_{obs}}{\sigma_t^2}\right) \\
\left(-\frac{\Delta A_{obs}}{\sigma_d^2}\right) \\
\left(\frac{v_{dn}\Delta A_{obs}}{\sigma_d^2}+\frac{v_{tn}\Delta B_{obs}}{\sigma_t^2}\right) \\
\left(-\frac{\Delta B_{obs}}{\sigma_t^2}\right) \\
\left(-\frac{\Delta A_{obs}(\mathcal{A}_{\max}+A)\delta_{\text{hyb},1}}{\sigma_d^2}\right) \\
\left(\frac{\Delta A_{obs}(\mathcal{A}_{\max}-A)\delta_{\text{hyb},2}}{\sigma_d^2}\right)
\end{pmatrix} \tag{58}
$$

We can rewrite these for programming purposes as

$$\begin{pmatrix} M_{00} & M_{01} & M_{02} & M_{03} & M_{04} & M_{05} & M_{06} & M_{07} \\ M_{10} & M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} & M_{17} \\ M_{20} & M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} & M_{27} \\ M_{30} & M_{31} & M_{32} & M_{33} & M_{34} & M_{35} & M_{36} & M_{37} \\ M_{40} & M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} & M_{47} \\ M_{50} & M_{51} & M_{52} & M_{53} & M_{54} & M_{55} & M_{56} & M_{57} \\ M_{60} & M_{61} & M_{62} & M_{63} & M_{64} & M_{65} & M_{66} & M_{67} \\ M_{70} & M_{71} & M_{72} & M_{73} & M_{74} & M_{75} & M_{76} & M_{77} \end{pmatrix} \begin{pmatrix} \delta\phi_d \\ \delta\phi_n \\ \delta\phi_t \\ \delta x_d \\ \delta x_n \\ \delta x_t \\ \delta\mu_1 \\ \delta\mu_2 \end{pmatrix} = \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{pmatrix} \tag{59}$$

where we use the following definitions:

$$\text{Useful per track } (l) \text{ quantities:} \quad \left\{ \begin{array}{ll} C_0 = 1/\sigma_d^2 & D_0 = 1/\sigma_t^2 \\ C_1 = v_{dn} C_0 & D_1 = v_{tn} D_0 \\ C_2 = A C_1 & D_2 = A D_1 \\ C_3 = B C_1 & D_3 = B D_1 \\ C_4 = B C_0 & D_4 = A D_0 \\ E_0 = v_{dn} C_3 + v_{tn} D_3 & \\ E_1 = v_{dn} C_2 + v_{tn} D_2 & \\ E_2 = D_2 - C_3 & \\ F_0 = (A + \mathcal{A}_{\max}) & F_1 = (A - \mathcal{A}_{\max}) \\ F_2 = F_0 C_0 \delta_{\text{hyb},1} & F_3 = F_1 C_0 \delta_{\text{hyb},2} \\ F_4 = v_{dn} F_2 & F_5 = v_{dn} F_3 \end{array} \right\}$$

$$\begin{array}{rcl} M_{00} & = & \sum_l (BE_0) \\ M_{01} & = & \sum_l (-BE_2) \\ M_{02} & = & \sum_l (-AE_0) \\ M_{03} & = & \sum_l (C_3) \\ M_{04} & = & \sum_l (-E_0) \\ M_{05} & = & \sum_l (D_3) \\ M_{06} & = & \sum_l (BF_4) \\ M_{07} & = & \sum_l (BF_5) \\ M_{11} & = & \sum_l (BC_4 + AD_4) \\ M_{12} & = & \sum_l (AE_2) \\ M_{13} & = & \sum_l (C_4) \\ M_{15} & = & \sum_l (-D_4) \\ M_{16} & = & \sum_l (BF_2) \\ M_{17} & = & \sum_l (BF_3) \\ M_{66} & = & \sum_l (F_0 F_2) \\ M_{77} & = & \sum_l (F_1 F_3) \end{array} \qquad \begin{array}{rcl} M_{22} & = & \sum_l (AE_1) \\ M_{23} & = & \sum_l (-C_2) \\ M_{24} & = & \sum_l (E_1) \\ M_{25} & = & \sum_l (-D_2) \\ M_{26} & = & \sum_l (-AF_4) \\ M_{27} & = & \sum_l (-AF_5) \\ M_{33} & = & \sum_l (C_0) \\ M_{34} & = & \sum_l (-C_1) \\ M_{36} & = & \sum_l (F_2) \\ M_{37} & = & \sum_l (F_3) \\ M_{44} & = & \sum_l (v_{dn} C_1 + v_{tn} D_1) \\ M_{45} & = & \sum_l (-D_1) \\ M_{46} & = & \sum_l (-F_4) \\ M_{47} & = & \sum_l (-F_5) \\ M_{55} & = & \sum_l (D_0) \end{array}$$

$$M_{14} = -M_{03} - M_{25}$$
$$M_{35} = M_{53} = M_{56} = M_{65} = M_{57} = M_{75} = M_{67} = M_{76} = 0$$

$$\begin{array}{lllll} M_{10} = M_{01} & M_{20} = M_{02} & M_{21} = M_{12} & M_{30} = M_{03} & M_{31} = M_{13} \\ M_{32} = M_{23} & M_{40} = M_{04} & M_{41} = M_{14} & M_{42} = M_{24} & M_{43} = M_{34} \\ M_{50} = M_{05} & M_{51} = M_{15} & M_{52} = M_{25} & M_{54} = M_{45} & M_{60} = M_{06} \\ M_{61} = M_{16} & M_{62} = M_{26} & M_{63} = M_{36} & M_{64} = M_{46} & M_{70} = M_{07} \\ M_{71} = M_{17} & M_{72} = M_{27} & M_{73} = M_{37} & M_{74} = M_{47} & \end{array}$$

$$\begin{array}{lcl} V_0 & = & \sum_l (\Delta A_{obs} C_3 + \Delta B_{obs} D_3) \\ V_1 & = & \sum_l (-\Delta A_{obs} C_4 + \Delta B_{obs} D_4) \\ V_2 & = & \sum_l (\Delta A_{obs} C_2 + \Delta B_{obs} D_2) \\ V_3 & = & \sum_l (-\Delta A_{obs} C_0) \\ V_4 & = & \sum_l (\Delta A_{obs} C_1 + \Delta B_{obs} D_1) \\ V_5 & = & \sum_l (-\Delta B_{obs} D_0) \\ V_6 & = & \sum_l (-\Delta A_{obs} F_2) \\ V_7 & = & \sum_l (-\Delta A_{obs} F_3) \end{array}$$

Only 39 running sums need to be calculated for each wafer while looping over all tracks to obtain the final matrix and vector for that wafer.

C Sign check on shifts

To make sure we add or subtract the shift quantities properly when correcting the database quantities, a quick test is performed here using simple shifts and Equations 1, 28, 30, and 33 to obtain the difference between the reconstructed $\Delta\vec{h}_{reco}$ and the ideal one with $\Delta\vec{h}' \equiv 0$.

A very simple translation shift in the $\hat{d}$ direction (allowing us to ignore the $\hat{n}$ and $\hat{t}$ quantities) gives:

$$\begin{array}{rclcl} \Delta\vec{h}_{reco} = \vec{h}_{hit} - \vec{h}_{track} & = & (A\hat{d} + \vec{x}) - \vec{h}_{track} & = & \Delta A\hat{d} \\ & & & = & -\delta x_d \hat{d} \\ \Delta\vec{h}' & = & (A\hat{d} + \vec{x}') - \vec{h}_{track} & \equiv & 0 \\ \Delta\vec{h}' - \Delta\vec{h}_{reco} & = & \vec{x}' - \vec{x} & = & \delta x_d \hat{d} \\ & & \vec{x}' & = & \vec{x} + \delta x_d \hat{d} \\ & & x'_d & = & x_d + \delta x_d \end{array}$$

This can be repeated for the other translations to give Equation 49.

For rotations, it is again easiest to take a simple case of a rotation about the $\hat{n}$ axis and extrapolate to the other rotations, this time ignoring $\hat{n}$ quantities:

$$\begin{array}{rclcl} \Delta\vec{h}_{reco} = \vec{h}_{hit} - \vec{h}_{track} & = & (A\hat{d} + B\hat{t}) - \vec{h}_{track} & = & \Delta A\hat{d} + \Delta B\hat{t} \\ & & & = & -B\delta\phi_n\hat{d} + A\delta\phi_n\hat{t} \\ \Delta\vec{h}' & = & (A\hat{d}' + B\hat{t}') - \vec{h}_{track} & \equiv & 0 \\ \Delta\vec{h}' - \Delta\vec{h}_{reco} & = & A(\hat{d}' - \hat{d}) + B(\hat{t}' - \hat{t}) & = & B\delta\phi_n\hat{d} - A\delta\phi_n\hat{t} \end{array}$$

$$\begin{array}{rclrcl} \hat{d}' - \hat{d} & = & -\delta\phi_n\hat{t} & \hat{t}' - \hat{t} & = & \delta\phi_n\hat{d} \\ \hat{d}' & = & \hat{d} - \delta\phi_n\hat{t} & \hat{t}' & = & \hat{t} + \delta\phi_n\hat{d} \end{array}$$

Again, this can be repeated for the other rotations, implying Equations 50-52.

Lastly, for drift velocity deviations, we use the simple case of an offset for hybrid 2 ($\delta_{\text{hyb},1} = 0$, $\delta_{\text{hyb},2} = 1$):

$$\begin{aligned}
\Delta\vec{h}_{reco} = \vec{h}_{hit} - \vec{h}_{track} &= (\mathcal{A}_{\max} - t\nu_2)\hat{d} - \vec{h}_{track} &&= \Delta A\hat{d} \\
& && = \delta\mu_2(\mathcal{A}_{\max} - A)\hat{d} \\
\Delta\vec{h}' &= (\mathcal{A}_{\max} - t\nu_2')\hat{d} - \vec{h}_{track} &&\equiv 0 \\
\Delta\vec{h}' - \Delta\vec{h}_{reco} &= -t(\nu_2' - \nu_2)\hat{d} &&= -\delta\mu_2(\mathcal{A}_{\max} - A)\hat{d} \\
& \nu_2' &&= \nu_2 + \delta\mu_2(\mathcal{A}_{\max} - A)/t \\
& &&= \nu_2 + \delta\mu_2(\nu_2) \\
& &&= \nu_2(1 + \delta\mu_2)
\end{aligned}$$

The same derivation can be done for hybrid 1, resulting in Equations 56 and 57.

D Alternative applications

D.1 Alignment without sensitivity to $\hat{n}$ shifts

If the tracks used in the alignment procedure do not cross the wafer planes with sufficient diversity of angles v_{dn} and v_{tn}, sensitivity of the χ^2 to the shifts $\delta x_n, \delta\phi_t, \delta\phi_d$ is lost. It may be better under such circumstances to constrain the wafers with zero values for these shifts than to allow the minimization the opportunity to walk the wafers in an attempt to find the best solution. Doing so is straightforward and involves reducing the matrix and vectors to only five rows and columns, removing anything associated with $\delta x_n, \delta\phi_t, \delta\phi_d, v_{dn}, v_{tn}$. Equation 59 becomes (needing only 16 running sums):

$$\begin{pmatrix} M_{00} & M_{01} & M_{02} & M_{03} & M_{04} \\ M_{10} & M_{11} & M_{12} & M_{13} & M_{14} \\ M_{20} & M_{21} & M_{22} & M_{23} & M_{24} \\ M_{30} & M_{31} & M_{32} & M_{33} & M_{34} \\ M_{40} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} \delta\phi_n \\ \delta x_d \\ \delta x_t \\ \delta\mu_1 \\ \delta\mu_2 \end{pmatrix} = \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} \tag{60}$$

$$\begin{aligned}
M_{00} &= \textstyle\sum_l (BC_4 + AD_4) & M_{11} &= \textstyle\sum_l (C_0) \\
M_{01} &= \textstyle\sum_l (C_4) & M_{13} &= \textstyle\sum_l (F_2) \\
M_{02} &= \textstyle\sum_l (-D_4) & M_{14} &= \textstyle\sum_l (F_3) \\
M_{03} &= \textstyle\sum_l (BF_2) & M_{22} &= \textstyle\sum_l (D_0) \\
M_{04} &= \textstyle\sum_l (BF_3) & M_{33} &= \textstyle\sum_l (F_0F_2) \\
& & M_{44} &= \textstyle\sum_l (F_1F_3)
\end{aligned}$$

$$M_{12} = M_{21} = M_{23} = M_{32} = M_{24} = M_{42} = M_{34} = M_{43} = 0$$

$$\begin{array}{ccccc}
M_{10} = M_{01} & M_{20} = M_{02} & M_{21} = M_{12} & M_{30} = M_{03} & M_{31} = M_{13} \\
M_{32} = M_{23} & M_{40} = M_{04} & M_{41} = M_{14} & M_{42} = M_{24} & M_{43} = M_{34}
\end{array}$$

$$\begin{aligned}
V_0 &= \textstyle\sum_l (-\Delta A_{obs} C_4 + \Delta B_{obs} D_4) \\
V_1 &= \textstyle\sum_l (-\Delta A_{obs} C_0) \\
V_2 &= \textstyle\sum_l (-\Delta B_{obs} D_0) \\
V_3 &= \textstyle\sum_l (-\Delta A_{obs} F_2) \\
V_4 &= \textstyle\sum_l (-\Delta A_{obs} F_3)
\end{aligned}$$

D.2 Ladder alignment in wafer-like coordinates

If we treat all wafers in a ladder as having the same alignment vectors $\hat{d}, \hat{n}, \hat{t}$ and consider $\vec{x}$ to be the position of the center of the ladder, then an alignment of the ladder position within those coordinates requires only a small modification from the individual wafer alignments. For the ladder, the measurement in the $\hat{t}$ direction becomes Z instead of B. The other necessary change is what to do about drift velocities. There are many choices, of which four seem useful: 1) ignore drift velocity deviations; 2) determine some average $\delta\mu_1, \delta\mu_2$ for the ladder; 3) same as option 2, but collapse all hybrids to one $\delta\mu$; and 4) expand the equations to include every hybrid on the ladder.

Options 2-4 are discussed below. The simple case of option 1 is detailed here as an example. Equation 59 becomes as follows (with 25 running sums):

$$\begin{pmatrix} M_{00} & M_{01} & M_{02} & M_{03} & M_{04} & M_{05} \\ M_{10} & M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{20} & M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{30} & M_{31} & M_{32} & M_{33} & M_{34} & M_{35} \\ M_{40} & M_{41} & M_{42} & M_{43} & M_{44} & M_{45} \\ M_{50} & M_{51} & M_{52} & M_{53} & M_{54} & M_{55} \end{pmatrix} \begin{pmatrix} \delta\phi_d \\ \delta\phi_n \\ \delta\phi_t \\ \delta x_d \\ \delta x_n \\ \delta x_t \end{pmatrix} = \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{pmatrix} \tag{61}$$

Changed per track (l) quantities: $\left\{ \; C_3 = ZC_1 \quad D_3 = ZD_1 \quad C_4 = ZC_0 \; \right\}$

$$\begin{array}{rcl} M_{00} & = & \sum_l (ZE_0) \\ M_{01} & = & \sum_l (-ZE_2) \\ M_{02} & = & \sum_l (-AE_0) \\ M_{03} & = & \sum_l (C_3) \\ M_{04} & = & \sum_l (-E_0) \\ M_{05} & = & \sum_l (D_3) \\ M_{11} & = & \sum_l (ZC_4 + AD_4) \\ M_{12} & = & \sum_l (AE_2) \\ M_{13} & = & \sum_l (C_4) \\ M_{15} & = & \sum_l (-D_4) \end{array} \qquad \begin{array}{rcl} M_{22} & = & \sum_l (AE_1) \\ M_{23} & = & \sum_l (-C_2) \\ M_{24} & = & \sum_l (E_1) \\ M_{25} & = & \sum_l (-D_2) \\ M_{33} & = & \sum_l (C_0) \\ M_{34} & = & \sum_l (-C_1) \\ M_{44} & = & \sum_l (v_{dn}C_1 + v_{tn}D_1) \\ M_{45} & = & \sum_l (-D_1) \\ M_{55} & = & \sum_l (D_0) \end{array}$$

$$M_{14} = -M_{03} - M_{25}$$
$$M_{35} = M_{53} = 0$$

$$\begin{array}{lllll} M_{10} = M_{01} & M_{20} = M_{02} & M_{21} = M_{12} & M_{30} = M_{03} & M_{31} = M_{13} \\ M_{32} = M_{23} & M_{40} = M_{04} & M_{41} = M_{14} & M_{42} = M_{24} & M_{43} = M_{34} \\ M_{50} = M_{05} & M_{51} = M_{15} & M_{52} = M_{25} & M_{54} = M_{45} & \end{array}$$

$$\begin{array}{rcl} V_0 & = & \sum_l (\Delta A_{obs} C_3 + \Delta Z_{obs} D_3) \\ V_1 & = & \sum_l (-\Delta A_{obs} C_4 + \Delta Z_{obs} D_4) \\ V_2 & = & \sum_l (\Delta A_{obs} C_2 + \Delta Z_{obs} D_2) \\ V_3 & = & \sum_l (-\Delta A_{obs} C_0) \\ V_4 & = & \sum_l (\Delta A_{obs} C_1 + \Delta Z_{obs} D_1) \\ V_5 & = & \sum_l (-\Delta Z_{obs} D_0) \end{array}$$

Option 2 is as in Appendix B with the Z-for-B switch. Option 3 may be preferable to option 2 if one chooses the average drift velocities approach because it reduces possible ambiguities in ΔA_{model} between drift velocities and certain rotations plus translations. For example,

$$\begin{aligned} \Delta A_{model} = & \quad -\delta\mu_1(\mathcal{A}_{\max} + A) & = \alpha + \beta A \\ \Delta A_{model} = & \quad -\delta x_d + v_{dn} A \delta\phi_t & \approx \alpha + \beta A \end{aligned}$$

when $v_{dn} \approx 1$, which is probably generally true for the available tracks. The two rows and columns associated with $\delta\mu_1, \delta\mu_2$ in Appendix B are replaced with a single set for $\delta\mu$ with

$$F_0 = (A - (\delta_{\mathrm{hyb},2} - \delta_{\mathrm{hyb},1})\mathcal{A}_{\max}) \tag{62}$$

$$F_2 = F_0 C_0 \tag{63}$$

F_1, F_3, F_5 become obsolete.

Option 4 is more complex as ladders on each barrel will have different numbers of parameters, depending upon the number of wafers on each ladder. Essentially, the columns and rows for $\delta\mu_1, \delta\mu_2$ are repeated for the hybrid pairs from each wafer w $(\delta\mu_{w1}, \delta\mu_{w2})$. The bottom right corner of the matrix (columns and rows beyond the first 6) is all zeros except for the diagonal elements:

$$\begin{array}{rcll} w & = & 0...n & \\ h & = & 1,2 & \\ F_{2,w} & = & F_0 C_0 \delta_{\mathrm{hyb},1} \delta_{\mathrm{wafer,w}} & \\ F_{3,w} & = & F_1 C_0 \delta_{\mathrm{hyb},2} \delta_{\mathrm{wafer,w}} & \\ F_{4,w} & = & v_{dn} F_{2,w} & \\ F_{5,w} & = & v_{dn} F_{3,w} & \\ M_{i,j} & = & 0 & \text{for} \quad i \neq j, i > 5, j > 5 \\ M_{i,i} & = & \sum_l (F_{h-1} F_{h+1,w}) & \text{for} \quad i = 5 + 2w + h \\ V_i & = & \sum_l (-\Delta A_{obs} F_{1+h,w}) & \text{for} \quad i = 5 + 2w + h \end{array}$$

Similarly, the cross terms in the bottom left and top right of the matrix are as in Appendix B, replacing F_{2-5} with $F_{2-5,w}$ as appropriate:

$$\begin{array}{rcll} M_{i,j} & = & M_{i,5+h}|_{F_{2-5} \to F_{2,w-5,w}} & \text{for} \quad i \leq 5, j = 5 + 2w + h \\ M_{i,j} & = & M_{5+h,j}|_{F_{2-5} \to F_{2w,-5,w}} & \text{for} \quad j \leq 5, i = 5 + 2w + h \end{array}$$