Notes on Special Relativity

Consider the four-vector $a \equiv (a^0, a^1, a^2, a^3)$ or $a = (a^0, \vec{a})$. The inner product of two four-vectors is invariant,

$$\boxed{a \cdot b \equiv a^0 b^0 - \vec{a} \cdot \vec{b}}.$$

The coordinate four-vector is $\boxed{x \equiv (t, \vec{r})}$ (units are such that $\hbar = c = 1$).

Similarly, $\boxed{dx = (dt, d\vec{r})}$

and $dx^2 \equiv dx \cdot dx = dt^2 - d\vec{r}^{\,2}$

or, $dx^2 = dt^2 (1 - \beta^2)$,

where $\boxed{\vec{\beta} \equiv \dfrac{d\vec{r}}{dt}}$. $(\vec{\beta} = \vec{v})$

We define $\boxed{\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}}$.

Then, $\boxed{dx^2 = \dfrac{dt^2}{\gamma^2} \equiv d\tau^2}$,

where $d\tau$ is the differential proper time interval. We define the velocity four-vector as

$$\boxed{u \equiv \frac{dx}{d\tau}}.$$

Then $\boxed{u = (\gamma, \gamma \vec{v})}$

and $\boxed{u^2 = 1}$.

We define the <u>momentum four-vector</u> as*

$$\boxed{P \equiv m u}.$$

Then $\boxed{P^2 = m^2 u^2 = m^2}$

and $\boxed{P = (m\gamma, m\gamma\vec{v})}$.

$\therefore$ We can write $\boxed{P = (E, \vec{P})}$,

where $\boxed{E = m\gamma}$

and $\boxed{\vec{P} = m\gamma\vec{v}}$.

Hence, $\boxed{\vec{v} = \frac{\vec{P}}{E}}$.

Now, $P^2 = m^2 = E^2 - \vec{P}^{\,2}$

$$\Rightarrow \boxed{E^2 = \vec{p}^{\,2} + m^2}.$$

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* For a particle of mass <u>m</u>.

We define the acceleration four
as

$$\boxed{a \equiv \frac{du}{d\tau}}\,.$$

Then, $$\boxed{\frac{dP}{d\tau} = ma}$$

is Newton's second law of mc
We identify the force four
as

$$\boxed{\mathcal{F} \equiv \frac{dP}{d\tau}}\,.$$

Consider the invariant, $\mathcal{F} \cdot dx$
We have

$$\mathcal{F} \cdot dx = \frac{dP}{d\tau} \cdot dx = \frac{dx}{d\tau} \cdot dP$$

$$= u \cdot dP = \frac{P}{m} \cdot dP$$

$$= \frac{1}{2m} d(P \cdot P)$$

$$= \frac{1}{2m} d(m^2)$$

$$\therefore \quad \boxed{\mathcal{F} \cdot dx \equiv 0}\,.$$

We can write, using Newton
law,

$$\boxed{\mathfrak{F} = \left(\gamma \frac{dE}{dt}, \gamma \vec{F}\right)},$$

where $\boxed{\vec{F} \equiv \frac{d\vec{P}}{dt}}$.

Then $\mathfrak{F} \cdot dX = \gamma\, dE - \gamma \vec{F} \cdot d\vec{r} = 0$

$$\Rightarrow \quad dE = \vec{F} \cdot d\vec{r}$$

or, $\boxed{E = \int \vec{F} \cdot d\vec{r}}$,

which is the <u>work-energy theorem</u>.

The <u>potential four-vector</u> of the electromagnetic field is defined by*

$$\boxed{A \equiv (\Phi, \vec{A})} \equiv A^{\mu} = g^{\mu\nu} A_{\nu}$$

and the field strengths are

$$\boxed{F^{\mu\nu} \equiv \frac{\partial A^{\mu}}{\partial X_{\nu}} - \frac{\partial A^{\nu}}{\partial X_{\mu}}}.$$

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* Here, $g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

We can write

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

Note that $\boxed{\frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)}$. $\left(\frac{\partial}{\partial x_\mu} \equiv \partial^\mu\right)$

It is straightforward to show that

$$\boxed{F^{\mu\nu} u_\nu = -\gamma\left(\vec{E}\cdot\vec{v},\ \vec{E} + \vec{v}\times\vec{B}\right)}$$

For a single particle with electric charge q in the presence of external fields $F^{\mu\nu}$, the Lorentz force is

$$\boxed{\mathcal{F}^{\mu}_{\text{Lorentz}} = -q F^{\mu\nu} u_\nu} \qquad \left(\text{or } \mathcal{F}^{\nu}_{\text{Lorentz}} = q u_\mu F^{\mu\nu}\right)$$

or,
$$\mathcal{F}_{\text{Lorentz}} = \gamma q\left(\vec{E}\cdot\vec{v},\ \vec{E} + \vec{v}\times\vec{B}\right).$$

Thus,

if we write $\mathcal{F}_{Lorentz} = \left(\gamma \frac{dE}{dt}, \gamma \vec{F}\right)$,
we have

$$\boxed{\begin{aligned} \vec{F} &= q\,(\vec{E} + \vec{v} \times \vec{B}) \\ \frac{dE}{dt} &= q\,\vec{E} \cdot \vec{v} \end{aligned}}\,.$$

Now we can write

$$\boxed{F^{\mu\nu} = \partial^{\nu} A^{\mu} - \partial^{\mu} A^{\nu}}\,,$$

which corresponds to the homogeneous Maxwell equations:

$$\left\{ \begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \end{aligned} \right\}$$

We define the current-density four-vector as

$$\boxed{J \equiv (\rho, \vec{J})}\,.$$

The continuity equation is $\boxed{\partial \cdot J = 0}$.

The inhomogeneous Maxwell equations,

$$\left\{ \begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= 4\pi \vec{J} \end{aligned} \right\}$$

can be written as

$$\boxed{\partial_\mu F^{\mu\nu} = -4\pi J^\nu} \; . \quad \left(\text{or } \partial_\nu F^{\mu\nu} = 4\pi J^\mu \right)$$