An alternative parametrization is to write

$$\boxed{\beta = \tanh \zeta} \;,$$

[Rapidity in high-energy physics is given symbol y]

where $\underline{\zeta}$ is called the rapidity. Thus,

$$\boxed{\begin{aligned} \gamma &= \cosh \zeta \\ \beta\gamma &= \sinh \zeta \end{aligned}} \;.$$

In terms of $\underline{\zeta}$, we can write

$$\boxed{\begin{aligned} x_0' &= x_0 \cosh \zeta - x_1 \sinh \zeta \\ x_1' &= -x_0 \sinh \zeta + x_1 \cosh \zeta \end{aligned}} \;.$$

These equations appear similar to those for a rotation of coordinates, but with hyperbolic functions instead of circular ones.

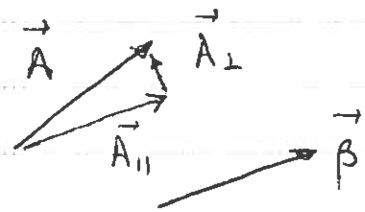
Four-Vectors

A physical quantity that transforms under Lorentz transformations as (x_0, x_1, x_2, x_3) is called a 4-vector. Thus, (x_0, x_1, x_2, x_3) is called the coordinate 4-vector. For an arbitrary 4-vector with components (A_0, A_1, A_2, A_3), where $\vec{A} = (A_1, A_2, A_3)$ is a 3-vector, the Lorentz transformation law is

$$\boxed{\begin{aligned} A_0' &= \gamma\left(A_0 - \vec{\beta}\cdot\vec{A}\right) \\ \vec{A}_{\parallel}' &= \gamma\left(\vec{A}_{\parallel} - \vec{\beta} A_0\right) \\ \vec{A}_{\perp}' &= \vec{A}_{\perp} \end{aligned}}$$

where $\vec{A}_{\parallel}$ $(\vec{A}_{\perp})$ is the component of $\vec{A}$ parallel (perpendicular) to the velocity $\vec{v} = c\vec{\beta}$.

We can write* $\quad \vec{A}_{\parallel} = \dfrac{(\vec{A}\cdot\vec{\beta})\vec{\beta}}{\beta^2}$



$$\vec{A}_{\perp} = \vec{A} - \vec{A}_{\parallel}$$

or, $$\vec{A}_{\perp} = \vec{A} - \frac{(\vec{A}\cdot\vec{\beta})\vec{\beta}}{\beta^2}.$$

Then, $$\begin{cases} \vec{A}_{\parallel}' = \gamma\left(\vec{A}_{\parallel} - \vec{\beta} A_0\right) \\ \vec{A}_{\perp}' = \vec{A}_{\perp} \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{(\vec{A}'\cdot\vec{\beta})\vec{\beta}}{\beta^2} = \gamma\left(\dfrac{(\vec{A}\cdot\vec{\beta})\vec{\beta}}{\beta^2} - \vec{\beta} A_0\right) \\ \vec{A}' - \dfrac{(\vec{A}'\cdot\vec{\beta})\vec{\beta}}{\beta^2} = \vec{A} - \dfrac{(\vec{A}\cdot\vec{\beta})\vec{\beta}}{\beta^2} \end{cases}$$

* $\vec{A}_{\parallel} \equiv (\vec{A}\cdot\hat{\beta})\hat{\beta}$

$$\therefore \quad \vec{A}' = \vec{A} - \frac{(\vec{A}\cdot\vec{\beta})\vec{\beta}}{\beta^2} + \gamma\left(\frac{(\vec{A}\cdot\vec{\beta})\vec{\beta}}{\beta^2} - \vec{\beta} A_0\right)$$

or,

$$\boxed{\vec{A}' = \vec{A} + \frac{(\gamma-1)}{\beta^2}(\vec{A}\cdot\vec{\beta})\vec{\beta} - \gamma\vec{\beta} A_0}$$

This equation gives the transformation from frame S to frame S', if the axes in S and S' remain parallel, but the velocity $\vec{\beta} c$ is an arbitrary direction.

Consider the transformation of the quantity,

$$A_0'^2 - \vec{A}'^2.$$

We have, [Homework Problem]

$$A_0'^2 - \vec{A}'^2 = \gamma^2 (A_0 - \vec{\beta}\cdot\vec{A})^2 - \left[\vec{A} + \frac{(\vec{A}\cdot\vec{\beta})\vec{\beta}}{\beta^2}(\gamma-1) - \gamma\vec{\beta} A_0\right]^2$$

$$= \gamma^2 [A_0^2 + (\vec{\beta}\cdot\vec{A})^2 - 2A_0\vec{\beta}\cdot\vec{A}] - [\vec{A}^2 + \frac{(\vec{\beta}\cdot A)^2}{\beta^2}(\gamma-1)^2 + \gamma^2\beta^2 A_0^2 + \frac{2(\vec{\beta}\cdot\vec{A})^2}{\beta^2}(\gamma-1) - 2\gamma A_0(\vec{\beta}\cdot\vec{A}) - 2\gamma(\gamma-1)(\vec{A}\cdot\vec{\beta})A_0]$$

$$\gamma^2(1-\beta^2)A_0^2 = A_0^2$$

$$\left[-2\gamma^2 + 2\gamma + 2\gamma(\gamma-1)\right] A_0(\vec{\beta}\cdot\vec{A}) = 0$$

$$\left[\gamma^2 - \frac{(\gamma-1)^2}{\beta^2} - \frac{2(\gamma-1)}{\beta^2}\right](\vec{\beta}\cdot\vec{A})^2$$

$$= \left[\beta^2\gamma^2 - (\gamma-1)^2 - 2(\gamma-1)\right]\frac{(\vec{\beta}\cdot\vec{A})^2}{\beta^2}$$

$$= \left[\gamma^2 - 1 - (\gamma-1)^2 - 2(\gamma-1)\right]\frac{(\vec{\beta}\cdot\vec{A})^2}{\beta^2} = 0$$

$$\therefore \quad \boxed{A_0'^2 - \vec{A}'^2 = A_0^2 - \vec{A}^2}$$

Clearly if $(B_0, \vec{B})$ and $(C_0, \vec{C})$ are 4-vectors, then $(B_0 + C_0, \vec{B} + \vec{C}) \equiv (A_0, \vec{A})$ is also a 4-vector. Thus, for any two 4-vectors,

$$(A_0' + B_0')^2 - (\vec{A}' + \vec{B}')^2 = (A_0 + B_0)^2 - (\vec{A} + \vec{B})^2$$

$$\Rightarrow (A_0'^2 - \vec{A}'^2) + (B_0'^2 - \vec{B}'^2) + 2(A_0'B_0' - \vec{A}'\cdot\vec{B}')$$

$$= (A_0^2 - \vec{A}^2) + (B_0^2 - \vec{B}^2) + 2(A_0B_0 - \vec{A}\cdot\vec{B})$$

Hence, we find that the "scalar product" of two 4-vectors is an invariant; i.e.,

$$\boxed{A_0'B_0' - \vec{A}'\cdot\vec{B}' = A_0B_0 - \vec{A}\cdot\vec{B}} \equiv A\cdot B$$

Proper Time and Time Dilation

Consider a system moving with an instantaneous velocity $\vec{u}(t)$ relative to some inertial system S. In a time interval dt, its position changes by $d\vec{r} = \vec{u}\,dt$. The corresponding invariant interval ds is

$$ds^2 = c^2 dt^2 - d\vec{r}^{\,2} = c^2 dt^2 (1-\beta^2),$$

where $\beta = u/c$. In the coordinate system S′ where the system is instantaneously at rest, the space-time increments are

$$dt' \equiv d\tau \quad \text{and} \quad d\vec{r}\,' = 0 .$$

Thus,
$$ds^2 = c^2 d\tau^2 ,$$

or,
$$\boxed{d\tau = dt\sqrt{1-\beta^2} = \frac{dt}{\gamma}} , \quad \left(\begin{matrix}\gamma \ge 1 \Rightarrow \\ d\tau \le dt\end{matrix}\right)$$

where $d\tau$ is a Lorentz invariant quantity. The time τ is called the proper time of the system. The blocked equation above expresses the phenomenon known as time dilation: A moving clock runs more slowly than a stationary clock.

Time dilation: $dt \ge d\tau$.

Relativistic Doppler Shift

Consider a plane wave of frequency ω and wave vector $\vec{k}$ in the inertial frame S. In the moving frame S', this wave will have a frequency ω' and wave vector $\vec{k}'$. The phase of the wave is an invariant quantity because the phase can be identified with the counting of wave crests in a wave train, an operation that must be the same in all inertial frames. Hence,

$$\phi = \omega t - \vec{k}\cdot\vec{r} = \omega' t' - \vec{k}'\cdot\vec{r}' .$$

This implies that the quantity $(\frac{\omega}{c}, \vec{k})$ forms a 4-vector like $(ct, \vec{r})$.
For light waves, $|\vec{k}| = \frac{\omega}{c}$ and $|\vec{k}'| = \frac{\omega'}{c}$.
Now,

$$\begin{cases} \frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \vec{\beta}\cdot\vec{k}\right) \\ k'_{\parallel} = \gamma\left(k_{\parallel} - \beta\frac{\omega}{c}\right) \\ k'_{\perp} = k_{\perp} \end{cases}$$

Note: In quantum mechanics, $(\frac{\hbar\omega}{c}, \hbar\vec{k})$ is the energy-momentum 4-vector of a photon w/ energy $\hbar\omega$ & momentum $\hbar\vec{k}$.

Let $\vec{\beta}\cdot\vec{k} \equiv \beta k \cos\theta = \beta\frac{\omega}{c}\cos\theta$

Then $\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \beta\frac{\omega}{c}\cos\theta\right)$,

or,
$$\boxed{\omega' = \gamma\omega(1-\beta\cos\theta)}$$

which is the <u>relatistic Doppler shift formula</u>.

Now
$$\begin{cases} k_{\parallel} = \dfrac{\vec{k}\cdot\vec{\beta}}{\beta} = k\cos\theta = \dfrac{\omega}{c}\cos\theta \\ k'_{\parallel} = \dfrac{\vec{k}'\cdot\vec{\beta}}{\beta} \equiv k'\cos\theta' = \dfrac{\omega'}{c}\cos\theta' \end{cases}$$

$$\Rightarrow \quad \frac{\omega'}{c}\cos\theta' = \gamma\left(\frac{\omega}{c}\cos\theta - \frac{\beta\omega}{c}\right)$$

$$\therefore \quad \boxed{\omega'\cos\theta' = \gamma\omega(\cos\theta - \beta)}$$

Also,
$$\begin{cases} k_{\perp} = k\sin\theta = \dfrac{\omega}{c}\sin\theta \\ k'_{\perp} = k'\sin\theta' = \dfrac{\omega'}{c}\sin\theta' \end{cases}$$

$$\Rightarrow \quad \boxed{\omega'\sin\theta' = \omega\sin\theta}$$

Therefore, we have
$$\boxed{\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}}$$