We have demonstrated that

$$\boxed{\begin{aligned} y' &= y \\ z' &= -z \end{aligned}}\ .$$

Similar arguments about the reciprocal nature of the coordinate systems in $\underline{S}$ and $\underline{S'}$ permit us to write

$$\boxed{\begin{aligned} x' &= Ax + Bt \qquad & t' &= Cx + Dt \\ x &= Ax' + Bt' \qquad & t &= Cx' + Dt' \end{aligned}}\ .$$

Thus, we have

$$\begin{cases} x = A(Ax + Bt) + B(Cx + Dt) \\ \quad = (A^2 + BC)x + (A + D)Bt \\ \\ t = C(Ax + Bt) + D(Cx + Dt) \\ \quad = (D^2 + BC)t + (A + D)Cx \end{cases}$$

Hence, we must have

$$\boxed{\begin{aligned} A^2 + BC &= 1 \qquad & (A + D)B &= 0 \\ D^2 + BC &= 1 \qquad & (A + D)C &= 0 \end{aligned}}\ .$$

We solve these equations to obtain

$$\boxed{\begin{aligned} A &= \pm\sqrt{1-BC} \\ D &= -A \end{aligned}}$$

For $u=0$, and origins $\underline{O}$ and $\underline{O'}$ coincident, we must have $x'=-x$, $t'=t$. This fixes the sign outside the square root above. We have now

$$\boxed{\begin{aligned} A &= -\sqrt{1-BC} \\ D &= +\sqrt{1-BC} \end{aligned}},$$

and only $\underline{B}$ and $\underline{C}$ remain to be determined.

We can write $\begin{cases} \Delta x' = A\,\Delta x + B\,\Delta t \\ \Delta t' = C\,\Delta x + D\,\Delta t \end{cases}$

$$\Rightarrow \quad \frac{\Delta x'}{\Delta t'} = \frac{A\,\dfrac{\Delta x}{\Delta t} + B}{C\,\dfrac{\Delta x}{\Delta t} + D}$$

In the limit $\Delta t \to 0$, we have

$$\boxed{v' = \frac{Av+B}{Cv-A}}$$

where $v = \frac{dx}{dt}$ and $v' = \frac{dx'}{dt'}$.

Let the moving point be $\underline{O'}$, so that $\underline{v = u}$ and $\underline{v' = 0}$, Then we obtain

$$0 = \frac{Au + B}{Cu - A},$$

which requires that $Au + B = 0$.

$$\therefore \; B = -Au = +u\sqrt{1 - BC}.$$

If we solve this equation for $\underline{C}$, we find

$$\boxed{C = \frac{1}{B} - \frac{B}{u^2}}.$$

Our remaining task is to determine $\underline{B}$.

We define the function $a = a(u)$ by

$$\boxed{B \equiv au}.$$

Then $\boxed{C = \frac{1 - a^2}{au}}$

$$BC = 1 - a^2$$

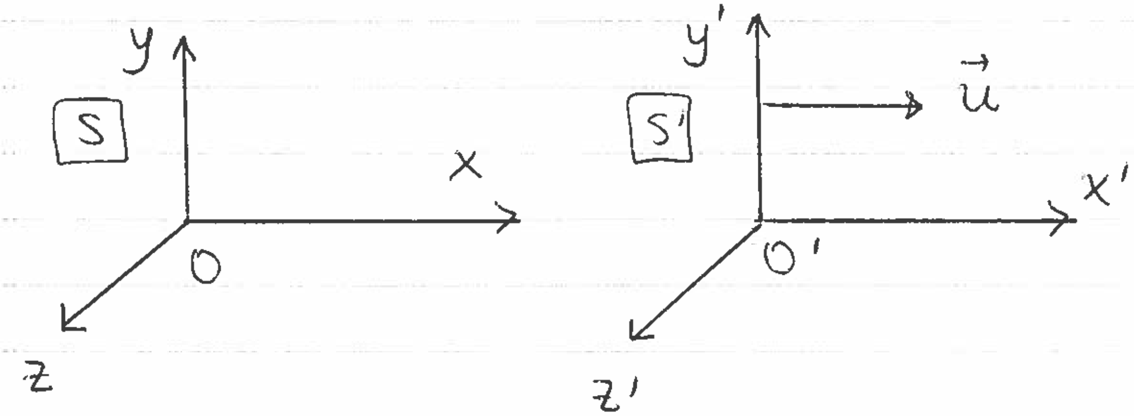
$$D = \sqrt{1 - BC} = \sqrt{a^2} \Rightarrow \boxed{D = a}$$

where $a(0) \equiv 1$. $\boxed{A = -a}$

Thus, we have

$$\boxed{\begin{aligned} x' &= -ax + aut \\ t' &= at + \left(\frac{1-a^2}{au}\right)x \end{aligned}}$$

For the remainder of the discussion, it will be convenient to rotate $\underline{x'}$ by $180°$ about the y'-axis, so that $\underline{x'}$ is parallel to $\underline{x}$, as shown below:



The transformation equations become:*

$$\boxed{\begin{aligned} x' &= ax - aut \\ t' &= at + \left(\frac{1-a^2}{au}\right)x \\ y' &= y \\ z' &= z \end{aligned}}$$

* We replace $\underline{x'}$ by $\underline{-x'}$.

We must determine the function $\underline{a(u)}$. To do this, we apply the principle of relativity in a manner that may be surprising, at first. We transform coordinates from $\underline{S}$ to a frame $\underline{S_1}$ moving with velocity $\vec{u}_1$ with respect to $\underline{S}$, and then from $\underline{S_1}$ to a frame $\underline{S_2}$ moving with velocity $\vec{u}_2$ with respect to $\underline{S_1}$ ($\vec{u}_1$ and $\vec{u}_2$ are parallel to $\underline{x}$). All coordinate transformations from one inertial frame to another must have the same form.

$$\text{Let} \qquad a_i \equiv a(u_i), \quad (i = 1, 2, 3)$$

where $\underline{\vec{u}_3}$ denotes the velocity of $\underline{S_2}$ with respect to $\underline{S}$.

$$\text{Let} \qquad K(u) \equiv \frac{1-a^2}{ua} \quad \text{and} \quad K_i \equiv K(u_i).$$

Trf. ~~from~~ S to S_1:
$$\begin{cases} x_1 = a_1 x - a_1 u_1 t \\ t_1 = a_1 t + K_1 x \end{cases}$$

Trf. from S_1 to S_2:
$$\begin{cases} x_2 = a_2 x_1 - a_2 u_2 t_1 \\ t_2 = a_2 t_1 + K_2 x_1 \end{cases}$$

Trf. from S to S_2:
$$\begin{cases} x_2 = a_3 x - a_3 u_3 t \\ t_2 = a_3 t + K_3 x \end{cases}$$

We have

$$x_2 = a_2(a_1 x - a_1 u_1 t) - a_2 u_2 (a_1 t + K_1 x)$$
$$t_2 = a_2(a_1 t + K_1 x) + K_2 (a_1 x - a_1 u_1 t)$$

or,
$$x_2 = a_2(a_1 - u_2 K_1) x - a_1 a_2 (u_1 + u_2) t$$
$$t_2 = a_1 (a_2 - u_1 K_2) t + (a_1 K_2 + a_2 K_1) x$$

Hence, we must have

$$a_3 = a_2(a_1 - u_2 K_1) = a_1 (a_2 - u_1 K_2)$$

$$\therefore \quad a_2 u_2 K_1 = a_1 u_1 K_2$$

Since $K_i = \dfrac{1-a_i^2}{a_i u_i}$, we find

$$\frac{1-a_1^2}{(a_1 u_1)^2} = \frac{1-a_2^2}{(a_2 u_2)^2}$$

$$\Rightarrow \quad \boxed{\frac{1-a^2}{(au)^2} = \text{const.} \equiv -\frac{1}{c_0^2}}$$

If we solve this equation for $\underline{a(u)}$, we obtain [with $a(0) = 1$]:

$$\boxed{a(u) = \left(1 - \frac{u^2}{c_0^2}\right)^{-1/2}}$$

Once we have found the numerical value of the universal constant c_0, our task of determining the transformation equations will have been completed. Experimentally, it turns out that

$$c_0 = c \simeq 3.00 \times 10^8 \ m/s .$$

Then we have

$$\boxed{a(u) = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} \equiv \gamma} .$$

The resulting transformation equations are

$$\boxed{\begin{aligned} x' &= \gamma(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{ux}{c^2}\right) \end{aligned}} .$$

These equations, known as the Lorentz transformation, form the basis of the special theory of relativity. Einstein derived these equations in a totally different manner (he used the principle of relativity plus the postulate that the speed of light has the same value in all inertial frames.)

Note that γ must be real, which requires that $\underline{u < c}$, which in turn implies that $\gamma \geq 1$.

The inverse transformation equations are obtained most easily by interchanging the roles of $\underline{S}$ and $\underline{S'}$; i.e., we interchange primed and unprimed coordinates and replace $\underline{u}$ by $-\underline{u}$. Then we obtain

$$\boxed{\begin{aligned} x &= \gamma(x' + ut') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{ux'}{c^2}\right) \end{aligned}}$$

Let $\begin{cases} x_0 \equiv ct \\ x_0' \equiv ct' \end{cases}$ $\quad \boxed{\beta \equiv \frac{u}{c}}$.

Then
$$\boxed{\begin{aligned} x_0' &= \gamma(x_0 - \beta x_1) \\ x_1' &= \gamma(x_1 - \beta x_0) \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}}$$

where now, $\boxed{\gamma = (1 - \beta^2)^{-1/2}}$.