Special Theory of Relativity

The original derivation of the formulas of the special theory of relativity made intimate use of electromagnetic theory. Such a derivation is at variance with the modern viewpoint that all physical phenomena are subordinate to the principles of the theory of relativity. Thus, new foundations of the theory of relativity have been developed, based upon obvious geometrical properties of space and time, and upon the tenet that physical laws are universal.

Suppose we have a laboratory located in some region of space far from all matter and radiation. In such an environment, space is believed to be homogeneous and isotropic. Space is homogeneous if the location of an event has no effect on the event; space is isotropic if all directions are equivalent. By event, we mean an occurrence at a point with coordinates (x, y, z) and at an instant of time t. The coordinates of an event are then defined by the four real numbers (x, y, z, t).

We submit that time is homogeneous: the choice of the zero of time can have no observable effect on the physical features of an event; only time intervals or differences are important.

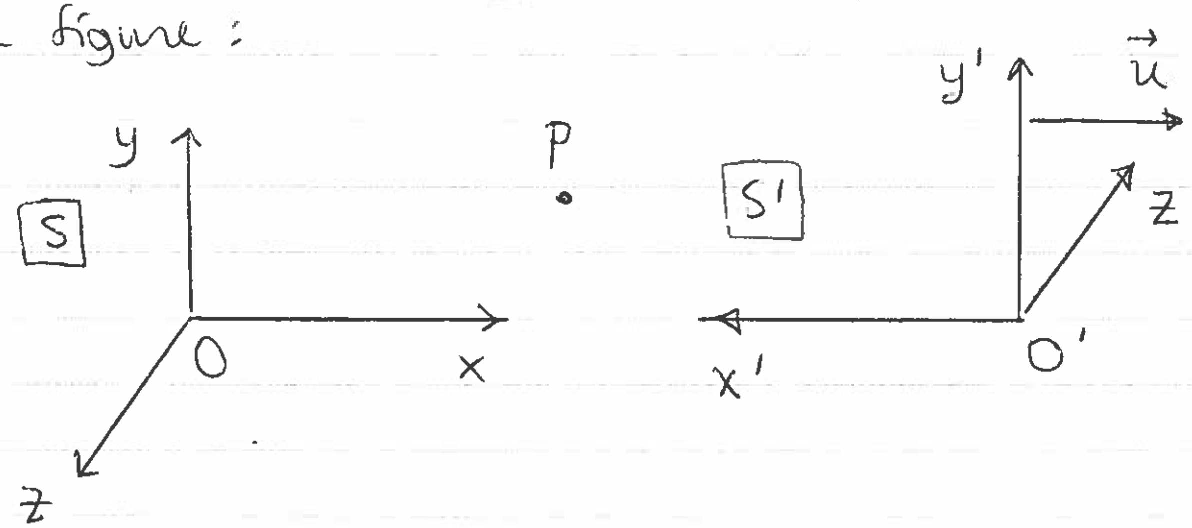
Because our laboratory is isolated from matter, and because forces between bodies diminish with increased separation, we can treat a test particle (a point mass) as a free particle. We refer to any coordinate frame in which the test particle's motion is uniform as an inertial frame. Here, uniform means in a straight line with constant velocity. We make the following postulate:

An observer in an inertial frame cannot detect his/her motion, relative to some other frame, by carrying out physical experiments within the confines of his/her own frame. I.e., the equations that describe physical laws have the same forms in all inertial frames. This is called the principle of relativity.

We take the <u>basic postulates of the special theory of relativity</u> to be:

(1) homogeneity of space and time;
(2) isotropy of space;
(3) principle of relativity.

Consider the coordinate systems in ~~which~~ the two inertial frames $\underline{S}$ and $\underline{S'}$ depicted in the figure:



The x and x' axes are antiparallel, the y and y' axes are parallel, and the z and z' axes are antiparallel. The point $\underline{P}$, an event, has coordinates (x, y, z, t) in $\underline{S}$ and (x', y', z', t') in $\underline{S'}$. At the time $t = t' = 0$, the origins $\underline{O}$ and $\underline{O'}$ coincide. Each coordinate system recedes from the other with velocity $\vec{u}$. The viewpoints of $\underline{S}$ and $\underline{S'}$ are <u>identical</u> because of the choice of coordinate systems.

We want to find mathematical expressions that relate (x, y, z, t) and (x', y', z', t').

Because of the homogeneity of space-time, $\Delta x'$, $\Delta y'$, $\Delta z'$, $\Delta t'$ must be independent of x, y, z, t. Thus, we have

$$\boxed{\begin{aligned} t' &= S_{00}t + S_{01}x + S_{02}y + S_{03}z + T \\ x' &= S_{10}t + S_{11}x + S_{12}y + S_{13}z + X \\ y' &= S_{20}t + S_{21}x + S_{22}y + S_{23}z + Y \\ z' &= S_{30}t + S_{31}x + S_{32}y + S_{33}z + Z \end{aligned}}$$

At $t = 0$ and $t' = 0$, the origins $\underline{O}$ and $\underline{O'}$ coincide. Thus, $x = y = z = 0$ at $t = 0$ and $x' = y' = z' = 0$ at $t' = 0$. Hence, we find

$$\boxed{T = X = Y = Z = 0}$$

At time $t = t' = 0$, a point lying on the y-axis must lie on the y'-axis; thus, the point $(0, y, 0, 0)$ must transform to $(0, y', 0, 0)$. This leads to

$$\begin{cases} t' = S_{02}\, y = 0 \\ x' = S_{12}\, y = 0 \\ z' = S_{32}\, y = 0 \end{cases}$$

which hold for all y only if

$$S_{02} = S_{12} = S_{32} = 0.$$

Similarly, a point $(0, 0, z, 0)$ on the z-axis must transform to $(0, 0, z', 0)$. Thus,

$$\begin{cases} t' = S_{03}\, z = 0 \\ x' = S_{13}\, z = 0 \\ y' = S_{23}\, z = 0 \end{cases}$$

which requires that

$$S_{03} = S_{13} = S_{23} = 0.$$

Furthermore, a point $(x, 0, 0, 0)$ on the x-axis must transform to $(x', 0, 0, t')$. Note that t' may <u>not</u> be zero. We have

$$\begin{cases} t' = S_{01}\, x \\ x' = S_{11}\, x \\ y' = S_{21}\, x = 0 \\ z' = S_{31}\, x = 0 \end{cases}$$

$$\Rightarrow \quad S_{21} = S_{31} = 0.$$

At all times, the origin $\underline{O}$ will have coordinates $y' = z' = 0$ in $\underline{S'}$. Thus,

$$y' = S_{20}t = 0$$
$$z' = S_{30}t = 0$$

$$\Rightarrow \quad S_{20} = S_{30} = 0 .$$

The transformation from (x, y, z, t) to (x', y', z', t') can now be written as

$$\boxed{\begin{aligned} t' &= S_{00}t + S_{01}x \\ x' &= S_{10}t + S_{11}x \\ y' &= S_{22}y \\ z' &= S_{33}z \end{aligned}}$$

Reciprocal Nature

$$y = S_{22}y'$$
$$z = S_{33}z'$$
$$\Rightarrow S_{22}^2 = S_{33}^2 = 1$$
$$\therefore \begin{cases} S_{22} = 1 \\ S_{33} = -1 \end{cases}$$

By rotating the coordinate system in $\underline{S}$ through $90°$ about $\underline{x}$, we can align the $\underline{y}$-axis with the direction formerly defined by $\underline{z}$ and vice versa. This means $\underline{y}$ and $\underline{z}$ must transform identically, which requires that

$$S_{22} = -S_{33} = g(u),$$

where $\underline{g(u)}$ is an unknown function of $\underline{u}$. The minus sign is required because $\underline{z}$ and $\underline{z'}$ point in opposite directions.

We introduce the notation:

$$\begin{cases} A = S_{11} \\ B = S_{10} \\ C = S_{01} \\ D = S_{00} \end{cases}$$

Thus, we obtain

$$\boxed{\begin{array}{ll} x' = Ax + Bt & t' = Cx + Dt \\ y' = g(u)\, y & \\ z' = -g(u)\, z & \end{array}}$$

The reciprocal nature* of the coordinate systems in $\underline{S}$ and $\underline{S'}$ permits us to write

$$\begin{cases} y = g(u)\, y' \\ z = -g(u)\, z' \end{cases}$$

Hence, we must have $[g(u)]^2 = 1$

or, $\boxed{g(u) = 1}$

We must choose the positive root because, in the case of no motion (i.e., $u=0$), we have $y' = y$ and $g(0) = 1$.

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* The reciprocal nature relies on the principle of relativity.