CLASSICAL ELECTRODYNAMICS II

Homework Set 7

April 13, 2018

1. Consider a point electric dipole rotating clockwise in the x-y plane with angular frequency ω. The time-dependent dipole moment can be written as

$$\mathbf{p}(t) = p_0(\hat{x}\cos\omega t + \hat{y}\sin\omega t) \ ,$$

so that $\mathbf{p}(0) = p_0\hat{x}$. Show that the corresponding dipole potential $\mathbf{A}(\mathbf{r}, t)$ agrees with the result obtained in homework set 5 for the vector potential in the radiation zone.

2. Consider two inertial frames S and S', which recede from each other with constant velocity $\mathbf{u} = \boldsymbol{\beta}c$. A 4-vector is defined as a set of four real numbers that transforms the same as the set of coordinates $(ct, x, y, z) = (ct, \mathbf{r})$. The Lorentz transformation equations for an arbitrary 4-vector $A = (A_0, \mathbf{A})$ can be written as

$$\begin{aligned} A_0' &= \gamma(A_0 - \boldsymbol{\beta}\cdot\mathbf{A}) \ , \\ \mathbf{A}_\parallel' &= \gamma(\mathbf{A}_\parallel - \boldsymbol{\beta}A_0) \ , \\ \mathbf{A}_\perp' &= \mathbf{A}_\perp \ , \end{aligned}$$

where $\mathbf{A} = \mathbf{A}_\parallel + \mathbf{A}_\perp$, with $\mathbf{A}_\parallel$ and $\mathbf{A}_\perp$ the components of $\mathbf{A}$ that are parallel and perpendicular to $\boldsymbol{\beta}$, respectively. It follows (as shown in class) that the 3-vector $\mathbf{A}$ transforms under a Lorentz transformation according to

$$\mathbf{A}' = \mathbf{A} + \frac{\gamma - 1}{\beta^2}\,(\mathbf{A}\cdot\boldsymbol{\beta})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}A_0 \ .$$

Use the general transformation equations for A_0 and $\mathbf{A}$ to show explicitly that

$$A_0'^2 - \mathbf{A}'^2 = A_0^2 - \mathbf{A}^2 \ .$$

That is, the quantity $A_0^2 - \mathbf{A}^2$ is a *Lorentz scalar* - it has the same value in *all* inertial frames.

3. An electromagnetic wave with frequency ω and propagation vector $\mathbf{k}$ impinges normally on a mirror, which is receding with velocity $\mathbf{v} = \beta c$

as measured in frame S. Determine the frequency of the reflected wave in frame S. (*Hint:* Consider the reflection from frame S$'$, the rest frame of the mirror.)

4. A coordinate system S' moves with a velocity $\mathbf{v}$ relative to another sytem S. In S' a particle has a velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$. Find the Lorentz transformation law for accelerations, and show that in the system S the component of acceleration parallel to $\mathbf{v}$ is

$$\mathbf{a}_{\parallel} = \frac{\left(1 - \dfrac{v^2}{c^2}\right)^{3/2}}{\left(1 + \dfrac{\mathbf{v} \cdot \mathbf{u}'}{c^2}\right)^3} \, \mathbf{a}'_{\parallel} \, .$$