CLASSICAL ELECTRODYNAMICS II

Homework Set 5

October 7, 2016

1. A plane polarized electromagnetic wave travelling in a dielectric with index of refraction n is reflected at normal incidence from the surface of a conductor with complex index of refraction $n(1 + \mathrm{i}\theta)$. Determine the phase change of the electric field of the reflected wave relative to that of the incident wave.

2. Flashlights and lasers produce narrow rays of illumination called *beams*. All beams have the property that their intensity profile falls off rapidly in the direction transverse to their direction of propagation. In addition, their intensity profile spreads out transversely as propagation proceeds. Another feature of beam-like waves is that they are *not transverse*. That is, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ generally have nonzero components parallel to the direction of propagation.

   Consider beam-like waves propagating in source-free space. Then in the Coulomb gauge, we may choose $\Phi = 0$ so that the fields are given by $\mathbf{E} = -\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$. The vector potential satisfies $\nabla \cdot \mathbf{A} = 0$ and
   $$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \ .$$
   Let $u(\mathbf{r}, t)$ be a scalar component of $A(\mathbf{r}, t)$ Then clearly,
   $$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \ .$$
   In the Lorentz gauge, Φ and the cartesian components of $\mathbf{A}$ satisfy the same equation but subject to the Lorentz condition,
   $$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \ .$$
   Let the z-axis be the direction of propagation. We may assume a trial solution in the form of a plane wave with a spatially modulated amplitude:
   $$u(\mathbf{r}, t) = \psi(\mathbf{r}_\perp, z)\mathrm{e}^{\mathrm{i}(kz-\omega t)} \ ,$$
   where $\mathbf{r}_\perp = x\hat{x} + y\hat{y}$ and $k = \omega/c$.

(a) Show that u is a solution of

$$\nabla_\perp^2 u + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0 \ ,$$

where

$$\nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2} \ .$$

(b) Next find the corresponding differential equation for ψ.

(c) We now assume that ψ changes very little in the z-direction over the wavelength $\lambda = 2\pi/k$. This assumption amounts to what is called the *paraxial approximation*:

$$k\frac{\partial \psi}{\partial z} \gg \frac{\partial^2 \psi}{\partial z^2} \ .$$

Use this approximation to obtain the *paraxial wave equation*:

$$-\frac{1}{2k}\nabla_\perp^2 \psi = \mathrm{i}\frac{\partial \psi}{\partial z} \ .$$

Note that if we replace $z \to t$, this equation has the form of a two-dimensional Schrödinger equation for a free particle with mass $m = \hbar k$.

(d) Show by direct substitution that the paraxial wave equation has a plane-wave solution of the form,

$$\psi(\mathbf{r}_\perp, z) = \mathrm{e}^{\mathrm{i}(\mathbf{q}_\perp \cdot \mathbf{r}_\perp)}\mathrm{e}^{-\mathrm{i}Ez} \ ,$$

and determine the "energy" E.

(e) The general solution of the paraxial wave equation may be constructed by making a linear superposition of plane waves:

$$\psi(\mathbf{r}_\perp, z) = \int d^2 q_\perp \ \hat{\psi}(\mathbf{q}_\perp) \ \mathrm{e}^{\mathrm{i}(\mathbf{q}_\perp \cdot \mathbf{r}_\perp)}\mathrm{e}^{-\mathrm{i}Ez} \ .$$

To obtain beam-like solutions, we must choose $\hat{\psi}(\mathbf{q}_\perp)$ such that $\psi(\mathbf{r}_\perp, z)$ falls to zero rapidly as $|\mathbf{r}_\perp|$ increases from zero. A suitable choice is the Gaussian weight function,

$$\hat{\psi}(\mathbf{q}_\perp) = w_0^2 \cdot \mathrm{e}^{-\frac{1}{4}w_0^2 q_\perp^2} \ .$$

Show that this choice allows the two-dimensional integral for $\psi(\mathbf{r}_\perp, z)$ to be factored into two identical one-dimensional integrals that can be evaluated in closed form. Carry out this integral using the identity,

$$\int_{-\infty}^{\infty} dx \; \mathrm{e}^{ax-bx^2} = \sqrt{\frac{\pi}{b}} \, \exp\left(\frac{a^2}{4b}\right) \; .$$

Use your result to determine the closed-form approximate solution of the original scalar equation in the form $u(\rho, z, t)$, where $\rho^2 = x^2 + y^2$.