CLASSICAL ELECTRODYNAMICS I

Homework Set 8

April 7, 2017

1. A charge distribution has azimuthal symmetry and can be described in spherical coordinates by the charge density $\rho(r, \theta)$.

   (a) Show that the traceless electric quadrupole moment tensor is diagonal for such a charge distribution; that is, show $Q_{ij} = 0$ for $i \neq j$.

   (b) Show that the diagonal elements Q_{11} and Q_{22} are proportional to Q_{33} and determine the proportionality constants.

2. The potential for a localized charge distribution can be written as a multipole expansion,

$$\Phi(\mathbf{r}) = \sum_{\ell=0}^{\infty} \Phi_\ell(\mathbf{r}),$$

   where

$$\Phi_\ell(\mathbf{r}) = \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{lm} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}},$$

   with

$$q_{lm} = \int Y^*_{lm}(\theta', \phi') r'^{\ell} \rho(\mathbf{r}') \, d\tau'.$$

   Here $\Phi_\ell(\mathbf{r})$ is the potential due to the 2^ℓ-pole term. Determine the behavior of the static electric field for a pure 2^ℓ-pole distribution under the transformation $\mathbf{r} \to -\mathbf{r}$.

3. Two concentric conducting spheres are centered at the origin. The inner sphere has radius a and carries charge $+Q$ while the outer sphere has radius b and carries charge $-Q$. The empty space between the spheres is filled with a dielectric with permittivity ϵ_1 for $z < 0$ ($\pi/2 < \theta < \pi$) and with a dielectric with permittivity ϵ_2 for $z > 0$ ($0 < \theta < \pi/2$).

(a) Determine $\mathbf{E}$ and $\mathbf{D}$ everywhere between the two spheres.

(b) Determine the free surface-charge density σ on the inner sphere.

(c) Determine the capacitance $C = Q/V$ of this system, where V is the potential difference between the inner and outer spherical conductors.