CLASSICAL ELECTRODYNAMICS I

Homework Set 6

March 4, 2015

1. A thin, flat, conducting, circular disk of radius R is located in the $z = 0$ plane with its center at the origin. The disk is maintained at a fixed potential V. In cylindrical coordinates (ρ, ϕ, z), the potential is $\Phi = \Phi(\rho, z)$ because of azimuthal symmetry. Consider the change of variables,

$$\rho = R \;\cosh u \cos v$$

$$z = R \;\sinh u \sin v \ ,$$

where (v, u, ϕ) define the *oblate spheroidal coordinate system.* (Note that $0 \leq u < \infty$ and $-\pi/2 \leq v \leq \pi/2$.) Surfaces of constant u are oblate spheroids and surfaces of constant v are hyperboloids. By solving the Laplace equation in oblate spheroidal coordinates, it may be shown that the electric potential of the disk is given by

$$\Phi = \frac{2V}{\pi} \tan^{-1}\left(\frac{1}{\sinh u}\right) \ .$$

   (a) Using the equation above for the potential, determine $\sigma(\rho)$, the charge density of the disk.

   (b) Determine the capacitance of the disk.

   (c) Using your equation for $\sigma(\rho)$, directly calculate the potential $\Phi(z)$ at any point on the z axis. Express your result in *closed form.* (Do *not* make use of the general equation for the potential in oblate spheroidal coordinates for your calculation!)

   (d) Expand your equation for $\Phi(z)$ as a power series in z. Then show that for $r > R$, the general solution for the potential in spherical coordinates is

$$\Phi(r, \theta) = \frac{2V}{\pi}\left(\frac{R}{r}\right)\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\left(\frac{R}{r}\right)^{2\ell} P_{2\ell}(\cos\theta) \ .$$

   (e) Find the potential for $r < R$.

2. A thin, flat, conducting, circular disk of radius R is located in the $z = 0$ plane with its center at the origin. As in problem 1, the disk is maintained at a fixed potential V. Use cylindrical coordinates (ρ, ϕ, z) to show that the general potential can be written as

$$\Phi(\rho, z) = \frac{2V}{\pi} \int_0^\infty \mathrm{e}^{-k|z|} J_0(k\rho) \frac{\sin kR}{k} \, dk \ .$$

Hint: Use the known potential on the z axis (from problem 1) and refer to a table of Laplace transforms.