QUANTUM MECHANICS

Homework Set 6

March 13, 2014

1. (a) Suppose that we have two complete sets of orthonormal basis states $\{|\psi_n\rangle\}$ and $\{|\phi_n\rangle\}$, which are related to each other by an operator S according to
$$|\phi_n\rangle = \sum_m S_{nm}|\psi_m\rangle \ .$$
Show that S must be a unitary operator:
$$SS^\dagger = S^\dagger S = I \ ,$$
where I is the identity operator. Thus, the transformation between the $\{|\psi_n\rangle\}$ and the $\{|\phi_n\rangle\}$ is a *unitary* transformation.

   (b) Now suppose that S is related to an operator K by
$$S = (I + \mathrm{i}K)(I - \mathrm{i}K)^{-1} \ .$$
Show that K must be hermitian if S is unitary.

2. We have seen that a plane wave, $\mathrm{e}^{\mathrm{i}kx}$, is not an acceptable wave function because it is not normalizable. However, consider the wave function,
$$\psi(x,0) = \begin{cases} A\mathrm{e}^{\mathrm{i}kx}, & \text{if } -\frac{2\pi n}{k} \le x \le \frac{2\pi n}{k}, \\ 0, & \text{otherwise,} \end{cases}$$
where n is an integer.

   (a) Show that $\psi(x,0)$ is normalizable and calculate A.

   (b) Calculate the corresponding momentum-space wave function $\Phi(p,0)$.

   (c) Is $\psi(x,0)$ an acceptable wave function? If not, explain why.

3. (a) Let us define a spatial displacement operator D_x by
$$D_x\psi(x) = \psi(x+a) \ .$$
Show that $D_x = \mathrm{e}^{\mathrm{i}a\hat{p}/\hbar}$, where $\hat{p}$ is the momentum operator. For this reason, $\hat{p}/\hbar$ is called the *generator* of translations in space.
Hint: Begin by expanding $\psi(x+a)$ in a Taylor series about $a = 0$.

(b) Now define a time displacement operator D_t by

$$D_t \psi(x,t) = \psi(x, t+t_0) \ .$$

Show that $D_t = \mathrm{e}^{-\mathrm{i}t_0\hat{H}/\hbar}$, where $\hat{H}$ is the Hamiltonian. For this reason, $-\hat{H}/\hbar$ is called the *generator* of translations in time. Note that both D_x and D_t are unitary operators.