QUANTUM MECHANICS

Homework Set 3

February 13, 2014

1. The ladder operators

$$a_{\pm} = \frac{1}{\sqrt{2m}} \left(\frac{\hbar}{\mathrm{i}} \frac{d}{dx} \pm \mathrm{i} m\omega x \right)$$

   generate new solutions to the Schrödinger equation for the one-dimensional harmonic-oscillator potential, but these new solutions are not correctly normalized. Thus $a_+\psi_n$ is proportional to ψ_{n+1} and $a_-\psi_n$ is proportional to ψ_{n-1}, but we'd like to know the precise proportionality factors.

   Show that apart from arbitrary phase factors,

$$a_+\psi_n = \sqrt{(n+1)\hbar\omega}\ \psi_{n+1}\ ,$$

$$a_-\psi_n = \sqrt{n\hbar\omega}\ \psi_{n-1}\ ,$$

   given that

$$\int_{-\infty}^{+\infty} f^*(a_{\pm}g)\ dx = \int_{-\infty}^{+\infty} (a_{\mp}f)^* g\ dx$$

   for any functions $f(x)$ and $g(x)$, provided that these integrals exist. (This identity follows because the ladder operators are hermitian operators.)

   Hint: Use the Schrödinger equation in the forms

$$(a_-a_+ - \frac{1}{2}\hbar\omega)\psi_n = E_n\psi_n\ ,$$

$$(a_+a_- + \frac{1}{2}\hbar\omega)\psi_n = E_n\psi_n\ ,$$

   where $E_n = (n + \frac{1}{2})\hbar\omega$.

2. Use the results from problem (1) and the othonormality of the eigenfunctions for the one-dimensional harmonic oscillator to calculate for the nth state ψ_n: (a) $\langle x\rangle$, (b) $\langle x^2\rangle$, (c) Δx, (d) $\langle p\rangle$, (e) $\langle p^2\rangle$, and (f) Δp. Also calculate (g) $\Delta x\Delta p$ and show that it is consistent with the uncertainty

principle. What value of n (if any) corresponds to the minimum uncertainty state? Finally calculate (h) $\langle T \rangle$ and $\langle V \rangle$ (the expectation values for the kinetic energy and potential energy, respectively).

Hint: Begin by solving for the position and momentum operators in terms of the ladder operators and then find all the results algebraically using your results from problem (1).

3. For the Delta-function potential, $V(x) = -\alpha\delta(x)$, the only bound state has the wave function

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}\, \mathrm{e}^{-\frac{m\alpha}{\hbar^2}|x|}$$

with energy

$$E = -\frac{m\alpha^2}{2\hbar^2} \; .$$

Calculate (a) $\langle x \rangle$, (b) $\langle x^2 \rangle$, (c) Δx, (d) $\langle p \rangle$, (e) $\langle p^2 \rangle$, and (f) Δp. Also calculate (g) $\Delta x \Delta p$ and show that it is consistent with the uncertainty principle.

Hint: You need to be especially careful in calculating $\langle p^2 \rangle$ because $d\psi(x)/dx$ is discontinuous at $x = 0$. It may be useful to work with the step function $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. Note also that $d\theta(x)/dx = \delta(x) = \delta(-x)$.