Sample Solutions — 1998 Classical Mechanics Homework Set IX

IX–A. Let θ be the angle of the string relative to the vertical. The kinetic and potential energies are

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad \text{and} \qquad V = -mgr\cos\theta\,.$$

The Lagrangian $L = T - V$ is

$$L = \tfrac{1}{2}m(v_0^2 + r^2\dot{\theta}^2) + mgr\cos\theta\,.$$

While the above is a valid expression for the Lagrangian, we must keep in mind that the string length r is not an independent generalized coordinate, but is constrained to be a simple linear function of time, i.e., $r = r_0 + v_0 t$.

In the Hamiltonian $H = p_i\dot{q}_i - L$, the implied sum term $p_i\dot{q}_i$ contributes only

$$\begin{aligned} p_\theta\dot{\theta} &= \frac{\partial L}{\partial\dot{\theta}}\dot{\theta} \\ &= mr^2\dot{\theta}^2 = \frac{p_\theta^2}{mr^2} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad H &= \frac{p_\theta^2}{mr^2} - \tfrac{1}{2}m\left(v_0^2 + \frac{p_\theta^2}{mr^2}\right) - mgr\cos\theta \\ &= \frac{p_\theta^2}{2mr^2} - \tfrac{1}{2}mv_0^2 - mgr\cos\theta \\ &= \frac{p_\theta^2}{2m(r_0+v_0t)^2} - \tfrac{1}{2}mv_0^2 - mg(r_0+v_0t)\cos\theta \end{aligned}$$

It can be seen that $\partial H/\partial t \neq 0$, so the Hamiltonian is not conserved. Moreover, $T + V = H + mv_0^2$, so the mechanical energy $T + V$ is not conserved. Physically, the nonconservation of both H and $T + V$ results from work being done on the mass m by some external force if the sign of v_0 is such that the string becomes shorter, and similarly, the mass does work if the string becomes longer.

IX–B. (1) Mathematically, the structure of this problem is identical to an earlier homework about a bead on a rotating hoop (II–D). In that problem, we considered a reference frame fixed to the hoop and accounted for the rotation using a centrifugal potential term. This time, we will follow a slightly different approach, using the $\boldsymbol{\omega}\times\mathbf{r}$ machinery developed in chapter 4.

There is just one generalized coordinate: θ, the deflection angle of the pendulum. (Note that now we are taking $\theta = 0$ at the 6 o'clock position, instead of at the 12 o'clock position as in homework solution II–D). Let us consider two Cartesian coordinate frames with their origins at the pivot point of the pendulum: one fixed, and the other attached to the hinge, i.e., rotating with angular frequency ω. Let (x, y, z) be the rotating system, with the direction of swing of the pendulum along x, and with z pointing up. Then the position of the mass m in the rotating system is

$$\mathbf{r} = (R\sin\theta,\ 0,\ -R\cos\theta)$$

In the fixed frame, the mass has an extra component of velocity

$$\begin{aligned}\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} &= (0,\ 0,\ \omega)\times(R\sin\theta,\ 0,\ -R\cos\theta) \\ &= (0,\ \omega R\sin\theta,\ 0) \qquad (1)\end{aligned}$$

Thus, the kinetic and potential energies in the fixed frame are

$$T = \tfrac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) \qquad \text{and} \qquad V = -mgR\cos\theta\,. \qquad (2)$$

Then $L = T - V$ and Lagrange's equation for θ gives us

$$\ddot{\theta} + \left(\frac{g}{R} - \omega^2\cos\theta\right)\sin\theta = 0\,.$$

(2) Clearly, the energy is a minimum at $\theta = 0$. For small deflections, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$; then the eq. of motion above reduces to

$$\ddot{\theta} + \left(\frac{g}{R} - \omega^2\right)\theta = 0\,.$$

As in the hoop problem, stable SHO-type motion requires the coefficient of the θ term to be positive, and this requires

$$\omega < \sqrt{\frac{g}{R}}\,,$$

in which case the solution is

$$\theta = \theta_0\cos(\omega' t + \delta)\,,$$

where $\omega' = \sqrt{(g/R) - \omega^2}$ is the angular frequency of oscillation, and θ_0 and δ are constants fixed by the initial conditions.

(3) Using eq. (1) to substitute for $m\mathbf{v}$ on the third line below, we can write the angular momentum about the z axis as

$$\begin{aligned}\ell &= \mathbf{r}\times\mathbf{p} \\ &= r_\perp p \\ &= (R\sin\theta)(m\omega R\sin\theta) \\ &= m\omega R^2\sin^2\theta\,.\end{aligned}$$

Then the torque supplied by the motor must be

$$\begin{aligned} N = \dot{\ell} &= 2m\omega R^2 \sin\theta \cos\theta\, \dot{\theta} \\ &= m\omega R^2 \dot{\theta} \sin 2\theta \\ &= -m\omega R^2 \omega' \theta_0 \sin(\omega' t + \delta) \sin[2\theta_0 \cos(\omega' t + \delta)] \,. \end{aligned}$$

(4) In the fixed frame, we use the result of part (1) to calculate the Hamiltonian:

$$\begin{aligned} H &= \frac{\partial L}{\partial \dot{\theta}} \dot{\theta} - L \\ &= mR^2 \dot{\theta}^2 - L \\ &= \tfrac{1}{2} mR^2 (\dot{\theta}^2 - \omega^2 \sin^2\theta) - mgR\cos\theta \\ &= T + V - mR^2\omega^2 \sin^2\theta \\ \text{and} \quad \frac{\partial H}{\partial t} &= 0 \,. \end{aligned}$$

In the rotating frame, we only have the first term in the kinetic energy given by eq. (2). However, in the rotating frame, the centrifugal force gives rise to a fictitious potential energy term V_{cent} (see homework II–D); then the total potential is

$$\begin{aligned} V &= V_{\text{cent}} + V_{\text{grav}} \\ &= -\tfrac{1}{2} mR^2 \omega^2 \sin^2\theta - mgR\cos\theta \,. \end{aligned}$$

(Don't forget the different choice for $\theta = 0$ relative to homework solution II–D, which flips the sign of V_{grav}). Now the Lagrangian in the rotating frame is

$$L_{\text{rot}} = \tfrac{1}{2} mR^2 (\dot{\theta}^2 + \omega^2 \sin^2\theta) + mgR\cos\theta \,,$$

and the Hamiltonian becomes

$$\begin{aligned} H_{\text{rot}} &= \frac{\partial L_{\text{rot}}}{\partial \dot{\theta}} \dot{\theta} - L_{\text{rot}} \\ &= mR^2 \dot{\theta}^2 - L_{\text{rot}} \\ &= \tfrac{1}{2} mR^2 (\dot{\theta}^2 - \omega^2 \sin^2\theta) - mgR\cos\theta \quad = \quad (T+V)_{\text{rot}} \\ &= H \,. \end{aligned}$$

(5) The Hamiltonian is identical in both cases above, and is conserved. In problem IX–A, work was continuously being done by or on the mass due to the changing length of the string. In the present problem, the situation might at first seem similar, because the motor exerts an external torque N [see part (3)]. However, observe that N oscillates so that, averaged over one or many cycles of the periodic motion, the effect of N on the energy of the pendulum cancels. The three terms in H represent the mechanical energy $(T+V)_{\text{rot}}$, which includes the centrifugal potential energy. This H is conserved and the work done by and on the motor does not arise in the rotating system.

In contrast, in the fixed system, the above-mentioned oscillating work done by/on the motor cannot be ignored, and it results in the non-conservation of $T+V$ as defined in this system.

IX–C. (1) The focusing element (the magnetic field of the quadrapoles) does not alter the number of particles in the beam. In the context of this problem, Liouville's Theorem then tells us that the phase space volume $\mathcal{V}$ occupied by the beam is not changed.

In this problem, two dimensions of the general six-dimensional phase space are irrelevant: the spatial and momentum dimensions along the direction of the particle beam. Thus we denote the initial phase space volume of the beam by

$$\mathcal{V}_1 \propto \pi r_1^2 \pi p_1^2 \,,$$

while at the focal point, the phase space volume of the beam is

$$\mathcal{V}_2 \propto \pi r_2^2 \pi p_2^2 \,,$$

with the same constant of proportionality.

As explained above, $\mathcal{V}_1 = \mathcal{V}_2$, so that

$$p_2 = p_1 \frac{r_1}{r_2}$$

i.e., squeezing the cross section of the beam into a narrower piece of configuration space necessarily makes the cross section of the beam in momentum space become larger.

(2) As a result of focusing down to a radius r_2, the beam will diverge at an angle α such that

$$\tan\alpha = \frac{p_2}{P} = \frac{p_1 r_1}{P r_2} \,.$$

At a distance d away, the maximum spatial radius of the beam will grow to $r(d) = r_2 + d\tan\alpha$, and we are asked to find d_{max} such that $r(d_{\text{max}}) = R$. Thus,

$$d_{\text{max}} = (R - r_2)\frac{P r_2}{p_1 r_1} \,.$$

(3) If loss of particles is not a consideration, the spatial cross section of a particle beam can be made smaller by passing through a collimator, without any obvious effect on the properties of the beam in momentum space. In terms of the notation of part (1), a collimator leads to $\mathcal{V}_1 > \mathcal{V}_2$, but Liouville's Theorem is not violated in this situation, since the number of particles also changes, keeping the phase space density D unchanged.