CANDIDACY EXAM – SPRING 2013

QUANTUM MECHANICS

Total: 100 Points

*** Attempt all the problems. There are several parts to each problem with unequal weights, shown in parentheses.

1. [35 points]

Imagine that a particle is subjected to a potential which allows it to move freely on the x-axis except that it may not be at \( x < -L/2 \) or \( x > +L/2 \).

[10 p] (a) Verify if the operators \( \hat{P} = -i\hbar \frac{d}{dx} \) and \( \hat{P}^2 \) are Hermitian by considering arbitrary wave functions, \( \psi \) and \( \phi \) (\( \psi \neq \phi \)), appropriate for the problem.

[16 p] (b) Obtain the the eigenfunctions and eigenvalues of \( \hat{P}^2 \) in this space? Comment on the parity of these eigenfunctions. Does \( \hat{P} \) have eigenfunctions in this space? Justify your answer.

[9 p] (c) What are the lowest three eigenvalues of \( \hat{P}^2 \) above? Show the three appropriately spaced eigenvalues, together with sketches of the corresponding three eigenstates, on a plot of eigenvalue spectrum.

Note: You may find it useful to recall the following formula for integration by parts:

\[
\int u dv = uv - \int v du
\]
2. [30 points]

A particle of mass $m$ experiences the following potential in 1-dimension (see sketch below):

$$V(x) = \begin{cases} 
\rightarrow 0, & \text{for } x \rightarrow -\infty \\
"something ghastly" \text{ (as in figure)}, & \text{for other } x \text{ values} \\
\rightarrow V_o > 0, & \text{as } x \rightarrow +\infty 
\end{cases}$$

Consider the particle to be incident from the left $(x \rightarrow -\infty)$ with definite momentum $p$, and energy $E > V_o$.

[10] (a) Find the form of the wavefunction of the particle for $x \rightarrow -\infty$ and $x \rightarrow +\infty$.

[8] (b) Find the probability current density for $x \rightarrow -\infty$ and $x \rightarrow +\infty$ using the wavefunctions found in part (a).

[12] (c) Using the continuity equation for probability and (b), derive a relation between the reflection and transmission coefficients.
3. [35 points]

Consider a spin 1/2 electron at rest in a uniform magentic field, $\vec{B} = B_z \hat{k}$. Then the Hamiltonian is given by:

$$H_0 = \frac{e}{m} B_z S_z$$

You may recall that in this case the eigenvalues are given by:

$$E_{a,b} = \mp \frac{e}{m} B_z \frac{\hbar}{2}$$

and the corresponding eigenvectors by:

$$\chi_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \chi_b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now turn on a perturbation $H_1$ in addition to $H_0$ given by:

$$H_1 = \frac{e}{m} B_x S_x$$

Find the variational upper bound on the ground state energy of the perturbed system using the following trial wavefunction:

$$\chi = (\cos \phi) \chi_a + (\sin \phi) \chi_b$$

where $\phi$ is the adjustable parameter. First, verify if $\chi$ is normalized.

Note: You may want to evaluate the needed matrix elements in the Dirac “bra-ket” notation. You may find the following relations useful:

(a) Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) Trigonometric relations:

$$\sin 2\phi = 2 \sin \phi \cos \phi$$

$$1 + \cos 2\phi = 2 \cos^2 \phi ; \quad 1 - \cos 2\phi = 2 \sin^2 \phi$$