Candidacy Exam, Spring 2012

Quantum Mechanics

Do all 4 questions. Some potentially useful results may be found at the end of the test.

(1) Consider a 2-dimensional potential $V(x,y) = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2y^2 + m\omega^2xy$. Note that the frequencies for the x and y oscillators, as well as the cross term, are the same.

(a) Write down the Schrodinger equation and change variables to $u = \frac{x+y}{\sqrt{2}}$ and $v = \frac{x-y}{\sqrt{2}}$.

You should get

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial u^2} - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial v^2} + \frac{1}{2}m\omega'^2u^2\psi = E\psi$$

where ω' is a new frequency (which you should identify).

(b) What are the normalized energy eigenfunctions (as functions of u and v) and eigenvalues? (You may use the notation φ_n for the normalized 1-dimensional harmonic oscillator eigenfunctions in your answer.)

(2) A hydrogen atom is in the state $\psi = \frac{\psi_{210}+\psi_{211}}{\sqrt{2}}$ (Notation: ψ_{nlm}).

Compute the expectation value $\langle \hat{L}_x^2 \rangle$.

(3) Some shorter questions.

(a) A 1-dimensional infinite square well of width L contains 120 non-interacting particles of mass m, each of spin 5/2. In the ground state, what is the energy of the highest single particle occupied energy level?

(b) Is the operator $\hat{O} = {}^{\partial}/_{\partial x}$ Hermitian? Prove your answer! (DO NOT ASSUME that the momentum operator is Hermitian in your proof!)

(c) Let φ_n be solutions of the time independent Schrodinger equation with energy eigenvalues ε_i. Suppose a system starts at t=0 in the state $\frac{\varphi_m+\varphi_n}{\sqrt{2}}$.

- *If you measured the energy, what values of energy might one get?*
- *Is energy conserved over time? Explain exactly what you mean by "conserved."*

(4) This question concerns a free electron with zero momentum. In your answers, refer to the magnetic moment of an electron as μ. Recall that the spin angular momentum operators may be written in terms of Pauli matrices as $\hat{S}_i = \frac{\hbar}{2}\sigma_i$.

(a) Assume that, at t=0, the spin points along $\hat{z}$. A magnetic field $\vec{B} = B\hat{x}$ is applied at t=0. Find the spin's wavefunction as a function of time.

The second part of this problem concerns time-independent properties of an electron in a field (this question is unrelated to that of part a). A magnetic field is applied along z: $\vec{B} = B\hat{z}$. We also apply a field-independent perturbation to the Hamiltonian $\widehat{\delta H} = a\hat{S}_x{}^3$ (a is a real constant).

(b) Use first order perturbation theory to find the shift in energy eigenvalues due to the perturbation for each eigenstates of the unperturbed Hamiltonian.

Some possibly useful expressions

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Angular momentum operators

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y,\ \hat{L}_- = \hat{L}_x - i\hat{L}_y$$

$$\hat{L}_+|lm> = \hbar\sqrt{l(l+1) - m(m+1)}\,|l\,m+1>$$

$$\hat{L}_-|lm> = \hbar\sqrt{l(l+1) - m(m-1)}\,|l\,m-1>$$

Eigenenergies for some potentials

Square well: $E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}$ (well width L, particle mass m)

Simple hydrogen atom: $E_n = \frac{m^2e^4}{2\hbar^2n^2}$