Candidacy Exam – Fall 2010

Quantum Exam

1.) Assume a particle of mass \( m \) in a one dimensional system with a potential \( V(x) = ax \) where \( a \) is a constant. Also assume a trial normalized wave function \( \psi \):

\[
\psi = \left( \frac{2b^4}{\pi} \right)^{1/4} e^{-bx^2},
\]

where \( b \) is a parameter \( \geq 0 \).

(10 points) a) Is the given \( \psi \) an eigenstate of the Hamiltonian \( H \)?

(20 points) b) For the given \( \psi \), calculate the expectation value of \( H \).

(10 points) c) Let \( E_0 \) be the notation for the ground state energy of the system and \( E_U \) be an upper bound, that is \( E_0 \leq E_U \). Determine the lowest value of \( E_U \) you can find using \( \psi \).

Possibly useful integrals:

\[
\int_0^\infty dx \ x^{2n} e^{-x^2} = \frac{1 \cdot (3) \cdots (2n-1) \sqrt{\pi}}{2^{n+1}}
\]

\[
\int_0^\infty dx \ x^{2n+1} e^{-x^2} = \frac{n!}{2}
\]

\[
\int_0^\infty dx \ e^{-x^2} = \frac{1}{2} \sqrt{\pi}
\]

2.) The spin-orbit operator is \( 2 \vec{L} \cdot \vec{S} \). Let \( \vec{J} = \vec{L} + \vec{S} \), where \( \vec{L} \) is orbital angular momentum, \( \vec{S} \) is spin, and \( \vec{J} \) is total angular momentum.

(15 points) a) Show that \( [S^2, \vec{L} \cdot \vec{S}] = [L^2, \vec{L} \cdot \vec{S}] = [J^2, \vec{L} \cdot \vec{S}] = 0 \).

(10 points) b) Derive the eigenvalues of the \( 2\vec{L} \cdot \vec{S} \) operator.
3.) In a representation where the eigenspinors of the $\hat{S}_z$ operator are the basis vectors $(\frac{1}{\sqrt{2}}, 0)$ and $(0, \frac{1}{\sqrt{2}})$, the $\hat{S}_z$ operator is
\[
\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
and the $\hat{S}_x$ operator is
\[
\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(10 points) a) Find the eigenvalues and normalized eigenspinors of $\hat{S}_x$.

(10 points) b) In an initial state of $\chi = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$, $\hat{S}_x$ is measured. What are the possible results, and their probabilities?

(10 points) c) Subsequently, $\hat{S}_z$ is measured. What are the possible results and their probabilities?

(5 points) d) What would have been the possible results and their probabilities if only $\hat{S}_z$ was measured (i.e. $\hat{S}_x$ had not been measured)?