## Quantum and Atomic Physics

1. (40 points) A particle of mass $m$ is moving in a one-dimensional box of length $L$ (infinite square well with the potential being zero in the interval $[0, L]$). At time $t = 0$ the state of the particle is known to be

$$\psi(x, t = 0) = \frac{1}{\sqrt{2}} [\psi_1(x) + \psi_2(x)] \ , \tag{1}$$

where $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$ is the $n^{th}$ energy eigenfunction with eigenvalue $E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2$ , for $n = 1, 2, 3, \ldots$ Introduce the notation $T = \frac{2\pi\hbar}{E_2 - E_1}$ .

(a) Starting from $\psi_1(x)$ and $\psi_2(x)$, (neatly) sketch $|\psi(x,t)|^2$ at the following times: $t = 0, \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}, T$ .

(b) Calculate the expectation value of the position of the particle $\langle x \rangle$ as a function of time.

(c) Calculate the expectation value of the momentum of the particle $\langle p \rangle$ as a function of time.

(Hint: There are a number of integrals to be carried out here. You may use without proof that

$$\int_0^L x \sin \frac{\pi x}{L} \sin \frac{2\pi x}{L} dx = -\frac{8L^2}{9\pi^2} \tag{2}$$

$$\int_0^L \cos \frac{\pi x}{L} \sin \frac{2\pi x}{L} dx = \frac{4L}{3\pi} \tag{3}$$

$$\int_0^L \sin \frac{\pi x}{L} \cos \frac{2\pi x}{L} dx = -\frac{2L}{3\pi} \ . \tag{4}$$

All other necessary integrals can be evaluated using simple arguments. Show your work and logic where such arguments are applied.)

2. (20 points) Consider a hydrogen-like atom with charge $Z$. Calculate the energy shift of the ground state caused by increasing the charge of the nucleus by one unit ($Z \longrightarrow Z+1$) in first order perturbation theory. Compare your answer to the exact result. Under what condition can the first-order perturbation calculation be considered a good approximation (within 1%)? (Recall that $E_n = -\dfrac{mZ^2e^4}{2\hbar^2 n^2}$ for the n$^{th}$ energy level of a hydrogen-like atom with charge $Z$, and that the average value of the Coulomb energy of an electron in the n$^{th}$ state of such an atom is $-\dfrac{mZe^4}{\hbar^2 n^2}$ , using the standard notation.)

3. (40 points) Spin and orbital angular momentum:

   (a) Calculate the eigenvalues and normalized eigenvectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. (You need to carry out the calculation; simply writing down the result is not sufficient.)

   (b) Suppose an electron is in the normalized spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. If $\hat{S}_y$ is measured, what is the probability of finding the value $\hbar/2$?

   (c) The total orbital angular momentum of a system is $\sqrt{56}\hbar$. Consider a measurement of the angle of the orbital angular momentum vector with a specified direction in space. What is the minimum angle such a measurement can yield?