Quantum Mechanics

① Work all four problems.

② Each is worth 25 points.

Possibly useful information

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

$$\int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8}$$

$$\int x \cos^2 x \, dx = \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}$$

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1. (25 pts.) For the potential

$$\begin{cases} v = \infty & x < \frac{-a}{2} \\ v = 0 & -\frac{a}{2} < x < \frac{a}{2} \\ v = \infty & x > +\frac{a}{2} \end{cases}$$

a) (15 pts.) Solve the Schrödinger equation for the eigenvalues and eigenfunctions.

b) (10 pts.) Consider now the perturbing potential Δv

$$\Delta v = \delta \qquad 0 < x < \frac{a}{2}$$

$$\Delta v = 0 \qquad \text{everywhere else}$$

Calculate the shifts in the eigenvalues for the first two states using perturbation theory. You may assume the normalization constant for the eigenfunctions is $\sqrt{2/a}$.

2. (25 pts.) Consider the two-dimensional isotropic harmonic oscillator with $V = \frac{1}{2} \alpha r^2 = \frac{1}{2} m \omega^2 r^2$

a) (15 pts.) Working in Cartesian coordinates, show that the solution is separable in the form , $\Psi_{n1,\,n2}(x,y) = \psi_{n1}(x)\,\psi_{n2}(y)$ where ψ_{n1} and ψ_{n2} are solutions to the one dimensional harmonic oscillator. You do not need to know the explicit forms of ψ_{n1} (or ψ_{n2}) but you do need to give an explicit expression for E_{n1n2}.

b) (5 pts.) What are the degeneracies of the first 5 energy eigenvalues? Include the quantum numbers as part of your answer.

c) (5 pts.) If the potential is changed to

$$V = \frac{1}{2} \alpha x^2 + \frac{1}{2} \beta y^2$$

What are the first 5 energy eigenvalues and their degeneracies if $\beta = 4\alpha$? Include the quantum numbers as part of your answer.

3. (25 pts.)

a) (5 pts.) What is the coordinate space operator for momentum $\vec{p}$?

b) (5 pts.) For a quantum state with a coordinate space wavefunction , $\psi(\vec{r}, t)$ write down the general expression for the expectation value $\langle\vec{p}\rangle$ in coordinate space.

c) (15 pts.) The corresponding momentum space wavefunction $\phi(\vec{p}, t)$ is defined as

$$\psi(\vec{r}, t) = \frac{1}{(2\pi\hbar)^{3/2}} \int \phi(\vec{p}, t)\, e^{i\vec{p}\cdot\vec{r}/\hbar}\, d^3p.$$

Substitute this in your answer to (b) and obtain an expression for $\langle\vec{p}\rangle$ as an expectation value in momentum space.

4. (25 pts.) The energy eigenvalues for the Coulomb potential are $E_n = -\frac{13.6\,\text{eV}}{n^2}$.

a) What are the degeneracies of these eigenvalues? (Explain from where your answer comes. Does your answer include degrees of spin freedom?)

These degeneracies are partially broken by various perturbations and residual interactions. For the following degeneracy-breaking corrections to the H atom energy levels, indicate, if you can, the order of magnitude (in eV) of the energy changes produced.

b) The Stark effect, due to an external electric field. Assume a field of 1000 V.

c) The Zeemann effect, due to an external magnetic field. Assume a field of 1 Tesla.

(a_o = Bohr radius ~ 0.5×10^{-10}m
μ_B = Bohr magneton ~ 10^{-4} eV/tesla)

d) What is the origin of "fine structure" in the H atom.

e) What is the origin of "hyperfine structure".