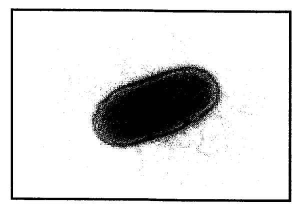
Classical Mechanics

Do all problems

1. (**35 pts.**) Consider an object of mass m moving in a 3-dimensional potential depending on the distance r from the origin as $U(r) = -a_1\left(\frac{r}{r_0}\right) + a_2\left(\frac{r}{r_0}\right)^2$ with $a_1, a_2 > 0$.

   a. (7pts.) Find the Lagrangian, the conjugate momenta, and from these, find the Hamiltonian H for this problem in Cartesian coordinates.
   b. (7 pts.) Show that the components of the angular momentum $\vec{\boldsymbol{L}}$ are conserved by evaluating the corresponding Poisson brackets [H, $\vec{\boldsymbol{L}}$]. Remember that the Poisson bracket of two functions u,v with respect to the canonical variables $\{q_i,p_i\}$ is defined by $[u,v] \equiv \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right)$.
   c. (7 pts.) Why can this problem be treated in strictly two dimensions, i.e., with all motion in a plane? List and justify any other quantities conserved in this problem.
   d. (7 pts.) Use the conservation laws to show that the equations of motion can be reduced to an effective one-dimensional problem with an effective one-dimensional potential. You may want to re-derive the Hamiltonian in an appropriate coordinate system.
   e. (7 pts.) Sketch the potential and the effective potential. Discuss the motion (bound, free, etc.) of this system for different choices of initial energies and positions. Identify on the sketch any conditions under which the motion will be circular and state (giving your reasons) whether such motion will be stable or unstable.

2. (**30 pts.**) In this problem you will explore a simple model for bacteria in motion. Bacteria typically live in water and other dense fluids. Certain bacteria move by pushing their pili (little hairs on their surface; see figure at right) through the water. As they move, they also experience a drag force from the fluid which can, when fluid inertia is negligible, be estimated as $\vec{\boldsymbol{F}}_{drag} = -c_d\vec{\boldsymbol{V}}$, where $\vec{\boldsymbol{V}}$ is the velocity of the bacteria.



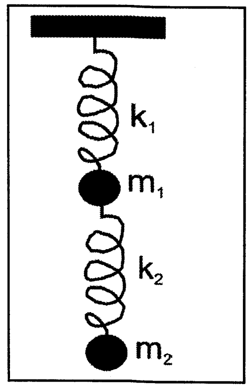
Now suppose that the bacterium starts at rest. The pili start beating, yielding a constant force $\vec{\boldsymbol{F}}_{applied}$ on the bacteria in the $\hat{\boldsymbol{x}}$ direction. The bacterium has mass $m \sim 10^{-15}$ kg.

   a. (5 pts.) Sketch qualitatively the total force, acceleration, and velocity of the bacteria as a function of time.
   b. (7 pts.) Find an expression for the velocity of the bacteria as a function of time in terms of c_d, m and $F_{applied}$.
   c. (6 pts.) Use dimensional analysis to find a relationship, within an unknown constant, between the coefficient of drag c_d, the average radius of the bacterium r, and the viscosity of the fluid η, considering that those and the velocity are the only relevant parameters of the system. (The units of viscosity are $[\eta] = \frac{kg}{s^2 \cdot m}$.)
   d. (7 pts.) Suppose that the final velocity of the bacterium is 10μm/s. Define an appropriate characteristic time for how long it takes to reach a significant fraction of that final velocity. Give an order of magnitude estimate for that time using the information in c.

Give an order of magnitude estimate for how far the bacterium moves in that time. You can take $\eta = 10^{-5}\frac{kg}{s^2 \cdot m}$ and $r = 1$ μm.

e. **(5 pts.)** Suppose that, after reaching the final velocity, the pili start beating to give a force perpendicular to the original direction of motion. Compare qualitatively the shape of the path followed by such a bacterium with that of a puck, also under a constant force perpendicular to the direction of motion, on a frictionless air table. Include a very rough sketch comparing the two types of motion.

3. **(35 pts.)** Consider two balls hanging vertically, in the direction of gravity, coupled by springs as shown in the figure. The upper (massless, lossless) spring has spring constant k_1, while the lower (similar) spring has spring constant $k_2 = \frac{3}{4}k_1$. The upper ball has mass m_1, while the lower ball has mass $m_2 = 3\ m_1$.



a. (10 pts.) Find the equations of motion for the two balls.

b. (15 pts.) Find the normal modes of oscillation.

c. (10 pts.) Suppose that the coupled springs are set in motion by holding the upper mass stationary at its equilibrium position with one hand and pulling the lower mass down from its equilibrium position by a distance d, and then letting both masses go together. Find the explicit equations for the subsequent motion.