Classical Mechanics

For full credit, answer any THREE questions. All questions carry equal credit. Labeled subparts within questions carry equal credit.

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1. A simple plane pendulum has a mass m at the end of a string of length r. While the pendulum is in motion, the length of the string is changed at a constant slow rate $v_0 = \dot{r}$. This is a famous problem discussed by Einstein, Lorentz and others at the 1911 Solvay Conference. Nowadays, the usual approach for finding the motion involves Adiabatic Invariants. However, you are not required to find the motion — just write down the Lagrangian and the Hamiltonian, and consider whether or not $T + V$ and H are conserved.

2. Consider a particle of mass m sliding without friction on the surface of a spherical bowl of radius R.
(a) Determine the Lagrangian in terms of polar and azimuthal angles θ and φ.
(b) Determine the generalized momenta p_θ and p_φ.
(c) Discuss cyclic coordinates and conserved quantities in the context of this example.
(d) If $\theta = \theta_0$ (a constant) at all times, find the velocity of the particle in terms of R and θ_0.

3. One side of earth's moon is always out of view on the earth, because the moon's orbital angular frequency is exactly the same as its rotational angular frequency. There are other examples in the solar system where an orbital angular frequency ω and a rotational angular frequency Ω, that one might expect to be uncorrelated, are in fact exactly synchronized (e.g., Mercury rotates exactly three times for every two of its orbits about the sun). The explanation is that ω and Ω generally start out being uncorrelated, but then tidal friction slowly dissipates the mechanical energy of the system in such a way as to synchronize ω and Ω. It can be predicted that the earth's moon will eventually (several billion years from now) become a geosynchronous satellite, by which time the earth's rotational period will have slowed to 47 of our present days.
(a) Is the radius of the moon's orbit increasing or decreasing? Outline the simplest possible quantitative reasoning, carefully stating any assumptions or approximations used.
(b) Consider a small moon of mass m in a circular orbit of radius r about a large planet of mass M and radius R. The moon's orbital angular frequency is ω. The easiest case to analyze is one where the *planet* rotates with frequency Ω and the

moon's rotation is neglected. Take the planet's rotation axis to be perpendicular to the plane of the moon's orbit. Show that the system stabilizes at $\Omega = \omega$.
Note: the above condition involves a first derivative. You are not required to take the second derivative or to find the condition for this stationary point to be a minimum.

4. Two identical springs (force constant k) hang from the ceiling a distance d apart. One edge of a homogeneous square sheet of material of mass M and sides d is attached to the ends of the springs. The system is constrained to move only in the vertical plane. Find the eigenfrequencies for small oscillations and describe each normal mode. Explain in words why the eigenfrequencies depend on k (which controls the spring potential energy), but not on g (which controls the gravitational potential energy).

You may assume the following formulae for moments of inertia about axes passing through the center of objects of mass m:
$I = 2mr^2/5$ for a solid sphere of radius r.
$I = md^2/6$ for a uniform square plate of side d, about an axis normal to the plate.