Candidacy Exam Fall 2011 Classical Mechanics

Q 1 [30pts]: A particle having electric charge q undergoes non-relativistic motion in the $x_1 - x_2$ plane under the influence of a static electromagnetic field where $\vec{E}$ is determined from a scalar potential $\phi(x_1, x_2)$ and $\vec{B}$ is determined from a vector potential $\vec{A}(x_1, x_2) = \frac{1}{2}(-x_2\hat{x}_1 + x_1\hat{x}_2)b$ where b is a constant. The Lagrangian is $L = T - V$ where $V = q\phi - \frac{q}{c}\vec{v} \cdot \vec{A}$, and $\vec{v}$ is the particle velocity.

- **(a)** Find the equation of motion for $x_i(t)$ in terms of $\phi, \vec{A}$.
- **(b)** Derive the Hamiltonian. Note, it is not $T + V$.
- **(c)** Show that the equation of motion of the particle is consistent with the Lorentz force obtained from $\vec{E}$ and $\vec{B}$.

Q 2 [20pts]: The π meson with rest mass M decays from rest to a muon (rest mass m and a neutrino (which can be considered to be massless). Find a relativistic formula for the energy of the muon in terms of the given masses.

Q 3 [30pts]: A particle of mass m starts from rest at the top of a solid hemisphere (of radius a) sitting on a table. The particle is allowed to slide (without friction) under the influence of gravity. After some time t_0, when its angular coordinate measured with respect to the vertical is θ_0, it will cease contact with the surface. Answer the following using the Lagrangian formalism with a single constraint, i.e., L is supplemented by $\lambda f(r, \theta)$ where λ is a Lagrange multiplier and $f = 0$ is the constraint.

- **(a)** For $\theta(t) < \theta_0$, find the equation of motion.
- **(b)** Find the angle θ_0.
- **(c)** Find the angular velocity at θ_0.
- **(d)** Find the equations of motion for $r(t), \theta(t)$ for $t > t_0$, but do not attempt to solve them.

[Hint: You may use $\int \ddot{\theta}\, d\theta = \int \dot{\theta}\, d\dot{\theta}$]

Q 4 [20pts]: A particle of mass m moves in a plane subject to a central potential $V(r) = -\frac{k}{r} + \frac{1}{2}br^2$. The Hamiltonian of the system in plane polar coordinates is

$$H = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + V(r) \equiv \frac{p_r^2}{2m} + V_{eff}(r)$$

where the radial component of the linear momentum is $p_r = m\dot{r}$ and the angular momentum $l = mr^2\dot{\theta}$ is constant.

- **(a)** From Hamilton's equations, obtain the equation of motion for r.
- **(b)** If the particle is in a stable circular orbit with $r = r_0$, argue that $\left(\frac{dV_{eff}}{dr}\right)_{r=r_0} = 0$ and use this condition to obtain l in terms of r_0, k, b, and m.
- **(c)** Consider now a small perturbation from the circular orbit such that $r - r_0$ is very small compared to r_0. Show that the particle undergoes small radial oscillations with angular frequency given by

$$\omega^2 = \frac{k}{mr_0^3} + \frac{4b}{m}.$$