Classical Mechanics

For full credit, answer ALL questions. Each question carries equal credit. Labeled subparts within questions carry equal credit.

---

1. One side of earth's moon is always out of view on the earth, because the moon's orbital period about the earth is exactly the same as its rotational period (both are about 28 days). There are other examples in the solar system where an orbital angular frequency ω and a rotational angular frequency Ω, that one might expect to be unrelated, are in fact exactly synchronized. The explanation is that ω and Ω generally start out being uncorrelated, but then tidal friction slowly dissipates mechanical energy in such a way as to synchronize ω and Ω. It can be predicted that the earth's moon will eventually (several billion years from now) become a geosynchronous satellite, i.e., by that time, the earth's rotational period will have lengthened to many times longer than our present 24-hour day, while the moon will orbit with that same period and will remain at a fixed point in the sky as seen from earth.
(a) Given the above information, is the moon's orbit moving closer to the earth or moving further away? Explain your reasoning quantitatively; you may neglect the rotation of the moon, and you should carefully state any further assumptions or approximations used.
(b) Consider a small moon of mass m in a circular orbit of radius r and orbital angular frequency ω about a large planet of mass M and principal moment of inertia I. The planet rotates with angular frequency Ω (the angular frequency vectors are parallel) and the moon's rotation can be neglected. Show that in a scenario where mechanical energy E is decreasing due to tidal dissipation, then $dE/d\Omega = 0$ if the two angular frequencies converge to the same value, but otherwise, $dE/d\Omega$ is non-zero.

2. A particle of mass m is balanced at the unstable equilibrium point on top of a smooth sphere of radius R. The sphere is not free to move. Use the Lagrange Multiplier method to find the angle from the vertical at which the particle, if given an initial velocity $\sqrt{gR/2}$, first loses contact with the sphere.

3. Consider a particle of mass m sliding without friction inside a spherical bowl of radius R.
(a) Write down the Lagrangian in terms of polar and azimuthal angles θ and φ.
(b) Determine the generalized momenta p_θ and p_φ.
(c) Discuss cyclic coordinates and conserved quantities in the context of this example. Remember to include time as a coordinate.
(d) If $\theta = \theta_0$ (a constant) at all times, find the velocity of the particle in terms of R and θ_0.

4. Three springs are fastened end-to-end in a straight line. The system has two identical masses, m, attached at each end of the middle spring, and the outside end of each outer spring is fixed rigidly. The middle spring has force constant k and the other two have force constants $k/3$. All springs are at their equilibrium length when the system is at rest, and neither gravity nor friction need be considered. Find the eigenfrequencies for small vibrations in a direction along the axis of the springs, and describe each normal mode of oscillation. If your initial choice of coordinates did not correspond to normal coordinates, describe a set of normal coordinates for this system.