Sample Solutions — Classical Mechanics Homework Set IV

A. *Goldstein Exercise 3–2:*
The possible ranges of E for each type of motion do not need to be given explicitly in terms of the constants a, k, l, and m.

The expression $V = (-k/r)e^{-ar}$ is familiar in atomic physics, where it is called the Screened Coulomb Potential. The $-k/r$ part represents the attractive Coulomb potential felt by a negative charge due to the nucleus of a neutral atom. At distances r comparable to the radius of the atom and beyond, the electrons' electric field cancels the nuclear electric field, hence the exponential term.

The equivalent one-dimensional potential is

$$V' = \frac{-k}{r}e^{-ar} + \frac{l^2}{2mr^2}$$

To get an idea about the shape of this potential, note first of all that $V' \approx l^2/2mr^2$ for both $ar \gg 1$ **and** $ar \ll 1$. Next, the stationary points tell us something about the "in-between" region:

$$\frac{\partial V'}{\partial r} = \frac{ak}{re^{ar}} + \frac{k}{r^2e^{ar}} - \frac{l^2}{mr^3} \tag{1}$$

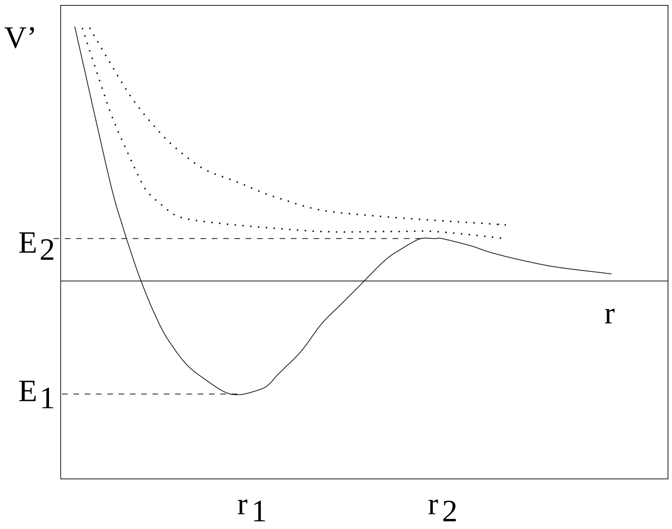
$$\frac{\partial V'}{\partial r} = 0 \quad \Rightarrow \qquad ar^2 + r = \frac{l^2}{mk}e^{ar}$$

This is a transcendental equation, and is best solved numerically. For some combinations of constants, the term on the right can be taken as constant and effectively we have a quadratic equation in r with two roots, r_1 and r_2. In this limit, it is straightforward to show that the smaller of these corresponds to a minimum in V' and the other to a maximum in V'. Hence, a V' shape resembling the solid curve in the figure below is possible.

The value of angular momentum l determines the radius r_1 where a stable orbit might be possible. The constant $1/a$ determines the radius where the screening term becomes important. If $1/a$ is made smaller and smaller in comparison to r_1, numerical experimentation reveals that the minimum in V' at r_1 becomes more shallow and shifts above $V' = 0$; for very small screening radius parameters $1/a$, the V' curves are dominated by the $l^2/2mr^2$ term for all r and resemble the dotted curves in the figure.

- For the dotted equivalent potentials, only unbounded orbits are possible.
- In the case of the solid curve,
  - for a particle with total energy $E > E_2 = V'(r_2)$, the motion is unbounded;

– for a particle with total energy $E = E_2$, an unstable circular orbit at radius r_2 is possible;
– for a particle with $E_2 > E > 0$ and $r > r_2$, its motion is unbounded;
– for a particle with $E_2 > E > E_1$ and $r < r_2$, it will move in a stable bounded orbit;
– and for a particle with total energy $E = E_1$, it will have a stable circular orbit at radius r_1.



Small radial oscillations about a stable circular orbit at r_1 take place according to (see Goldstein's Appendix A)

$$\frac{1}{r} = \frac{1}{r_1} + A\cos\beta\theta \qquad \text{where} \qquad \beta^2 = 3 + \frac{r}{f}\frac{df}{dr}\Big|_{r=r_1}$$

The orbiting body undergoes one cycle of radial oscillation for $\beta\Delta\theta = 2\pi$. The period $\tau_{\rm osc}$ for this oscillation is related to the body's tangential speed v in the circular orbit by

$$\tau_{\rm osc} = \frac{r_1\Delta\theta}{v} = \frac{2\pi r_1}{v\beta}\,.$$

For any circular orbit, $mv^2/r = f$, so we can write

$$\tau_{\rm osc}^2 = \frac{4\pi^2 r_1 m}{\beta^2 f} = \frac{4\pi^2 m}{3f(r_1)/r_1 + (df/dr)_{r=r_1}}$$

Finally, we need to substitute $f = -dV/dr = (a + 1/r)V$ (see eq. (1) above) and

$$\frac{df}{dr} = -(a + \frac{1}{r})f - \frac{V}{r^2} = -\left(\frac{2}{r^2} + a^2 + \frac{2a}{r}\right)V$$

In chapter 6, we consider small oscillations in more detail.

B. *Goldstein Exercise 3–14:*

We begin from the following eq. for $u = 1/r$ [see lecture notes or Goldstein (3–34b)], which holds for any central potential V:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{dV}{du}$$

We are given $V = -ku + hu^2$, and so $dV/du = -k + 2hu$. Our differential eq. for the orbit thus becomes:

$$\frac{d^2u}{d\theta^2} + u(1 + \frac{2mh}{l^2}) - \frac{mk}{l^2} = 0$$

The above takes on the form of the familiar Simple Harmonic Oscillator equation $d^2y/d\theta^2 + \omega^2 y = 0$ if we identify

$$y = u - \frac{mk}{l^2 + 2mh} \qquad \text{and} \qquad \omega^2 = 1 + \frac{2mh}{l^2}$$

The SHO solution is $y = y_0 \cos(\omega\theta - \theta_0)$, which we can now write in the form:

$$\frac{1}{r} = \frac{mk}{l^2 + 2mh}[1 + e' \cos(\omega\theta - \theta_0)] \tag{2}$$

We don't need to worry about writing down the explicit form for the eccentricity e'; as we will see later, it isn't needed in this problem. If we set $h = 0$, notice that the above reduces (as required) to the standard Kepler orbit formula:

$$\frac{1}{r} = \frac{mk}{l^2}[1 + e \cos(\theta - \theta_0)]$$

Eq. (2) has the form of our standard Kepler orbit if we transform to a coordinate system where $\theta' = \omega\theta$. What is the physical meaning of this transformation?

The condition to return to the same point on the orbit in the transformed coordinate is that θ' should change by an amount $\Delta\theta' = 2\pi$, or that $\Delta\theta = 2\pi/\omega$. The quantity $(\Delta\theta' - \Delta\theta)$ is the difference in phase between one complete orbit in the primed coordinates (where the effect of the h term in the potential is cancelled) and one complete orbit in the original coordinate system. In other words, the primed system precesses in step with the orbit described by Eq. (2), and in this primed system, the orbit looks like a normal Keplerian ellipse. Since precession causes a change of phase angle $(\Delta\theta' - \Delta\theta)$ in one orbit, over a time period τ, the rate of precession is

$$\dot{\Omega} = \frac{\Delta\theta' - \Delta\theta}{\tau} = \frac{2\pi - 2\pi/\omega}{\tau} = \frac{2\pi(1 - 1/\omega)}{\tau}.$$

Now we use the fact that h is small to write $1/\omega \approx 1 - mh/l^2$, and so we get

$$\dot{\Omega} = \frac{2\pi mh}{l^2\tau}$$

From lecture notes or Goldstein eq. (3-63), we know that $l^2 = mka(1-e^2)$ for a Kepler potential; for $h \neq 0$, eq. (2) above implies that $l^2 + 2mh = mka(1-e'^2)$. For the purpose of evaluating $\dot{\Omega}$, the added $2mh$ term is completely negligible, so we end up with

$$\dot{\Omega} = \frac{2\pi\eta}{(1-e'^2)\tau} = 1.9 \times 10^{-6} \;\; \text{radians/year}$$

or about 40 seconds of arc per century. (Note that there are 60×60 seconds of arc in a degree.) The total observed rate of precession of Mercury's orbit is 574 seconds of arc per century, but most of this can be explained by the gravitational force from other planets. The h term in the Sun's potential was orginally intended to describe the small remaining $43''$ per century which cannot be explained by Newtonian mechanics. In fact, general relativity accurately accounts for this deficit in the rate of precession of Mercury's perihelion.

C. *Goldstein Exercise 3–16:*

We begin from Kepler's 3rd law:

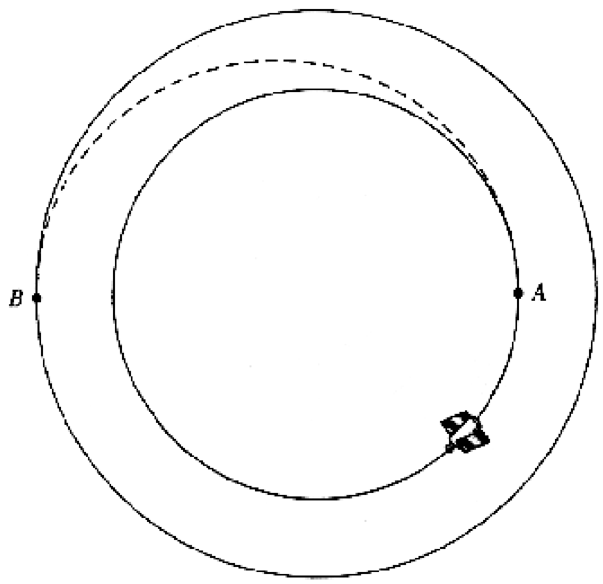
$$\tau = \frac{2\pi a^{3/2}}{\sqrt{G(M+m)}} \approx \frac{2\pi \langle r \rangle^{3/2}}{\sqrt{GM}} \; .$$

This is a good approximation in the present case; the small eccentricity of the orbits of both Moon and Earth allow us to write the mean radius $\langle r \rangle \approx a$, and $M + m \approx M$ because the Earth is very much more massive than the Moon, and also the Sun is very much more massive than the Earth. Thus

$$\tau^2_{\text{Earth}} \propto \frac{\langle r \rangle^3_{\text{Earth}}}{M_{\text{Sun}}} \qquad \text{and} \qquad \tau^2_{\text{Moon}} \propto \frac{\langle r \rangle^3_{\text{Moon}}}{M_{\text{Earth}}} \; ;$$

$$\frac{M_S}{M_E} = \frac{\langle r_E \rangle^3 \tau_M^2}{\langle r_M \rangle^3 \tau_E^2} = 3.4 \times 10^5 \; .$$

D. *In homework I-D, we considered the problem of a rocket which left the earth by accelerating straight up (antiparallel to* $\mathbf{g}$*) until it reached earth escape velocity. A more energy-efficient approach to interplanetary travel is to coast from one orbit to another. Suppose that a space vehicle is already in orbit around the sun in the same orbit as the earth, semimajor axis* $a_E = 150$ *million km, and is far enough from the earth that the Sun's gravity dominates. Using the approximation that the eccentricity of the orbit of both Earth and Mars is roughly zero, and taking the orbits to be coplanar, calculate the minimum increase in velocity needed to enter an elliptic orbit which intersects the orbit of Mars,* $a_M = 228$ *million km. Also calculate the duration of the journey. Express answers in terms of* a_E*,* a_M*, and periods* τ*, and also give numerical values.*



There are many elliptic orbits which intersect the orbit of both Earth and Mars. However, the orbit is made unique by requiring that the energy be a minimum (or equivalently, that the increase in velocity upon leaving Earth orbit be a minimum). Then the required elliptic orbit must intersect Earth orbit at its closest point to the Sun (point A in the diagram) and intersect Mars orbit at its farthest point from the Sun (point B in the diagram). This elliptic transfer orbit thus has a semimajor axis $a_t = (a_E + a_M)/2$.

Let v_0 be the orbital speed in the Earth's orbit, and let Δv be the extra speed needed to enter the transfer orbit. We use eq. (3–61): $a = -k/2E$ to write

$$E_{\text{vehicle}} = -\frac{k}{a_E + a_M} = \frac{m(v_0 + \Delta v)^2}{2} - \frac{k}{a_E}$$

$$(v_0 + \Delta v)^2 = \frac{2k}{ma_E}\,\frac{a_M}{a_E + a_M}\,.$$

In the vehicle's initial circular orbit, the radial part of the kinetic energy is zero, the centrifugal force is $mv_0^2/a_E = k/a_E^2$ and so

$$v_0 + \Delta v = v_0 \left(\frac{2a_M}{a_E + a_M}\right)^{1/2} .$$

This is almost the expression we were asked for, except we need to substitute $v_0 = 2\pi a_E/\tau_E$, where τ_E is the Earth's period, to get

$$\Delta v = \frac{2\pi a_E}{\tau_E}\left[\left(\frac{2a_M}{a_E + a_M}\right)^{1/2} - 1\right] = 2.9 \text{ km/s}$$

Notice that this would typically require one modern rocket stage to burn less than half its fuel.

The duration T of the journey can be most easily obtained from Kepler's 3rd law, which gives us τ_t, the period of the transfer orbit:

$$T = \frac{\tau_t}{2} = \left(\frac{a_E + a_M}{2a_E}\right)^{3/2} \frac{\tau_E}{2} = 0.71 \text{ years}$$