Sample Solutions — Classical Mechanics Homework Set II

A. *Goldstein Exercise 1–14:*

We are given $L' = L + \dot{F}$, where $F = F(q_1, ...q_n, t)$. Therefore,

$$\dot{F} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}$$

$$\frac{\partial \dot{F}}{\partial \dot{q}_i} = \frac{\partial F}{\partial q_i} \quad \text{from above;} \qquad \text{also} \quad \frac{\partial \dot{F}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right)$$

Inserting $L = L' - \dot{F}$ into Lagrange's eq.:

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial F}{\partial q_i}\right) - \frac{\partial L'}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right) = 0$$

Because the 2nd and 4th terms cancel, we have shown that L' also satisfies Lagrange's eq.

B. *Goldstein Exercise 1–17:*
In addition, comment on the physical meaning of the two equations you obtain, and use them to describe (in words) at least two different special cases of the possible motion.

For diagrams, and setup of Lagrangian, see lecture notes.

$$L = \frac{mr_0^2}{2}(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 + \frac{2g}{r_0}\cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr_0^2\dot{\theta}\,; \qquad \frac{\partial L}{\partial \theta} = mr_0^2(\sin\theta\cos\theta\,\dot{\varphi}^2 - \frac{g}{r_0}\sin\theta)$$

$$\frac{\partial L}{\partial \dot{\varphi}} = mr_0^2\sin^2\theta\,\dot{\varphi}\,; \qquad \frac{\partial L}{\partial \varphi} = 0$$

Lagrange's eq. for θ and φ $\Rightarrow$

$$mr_0^2\,(\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 + \frac{g}{r_0}\sin\theta) = 0 \quad \text{and} \quad mr_0^2\sin\theta\,(\sin\theta\,\ddot{\varphi} + 2\cos\theta\,\dot{\theta}\dot{\varphi}) = 0$$

Therefore, the equations of motion in simplest form are:

$$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 + \frac{g}{r_0}\sin\theta = 0 \quad \text{and} \quad \ddot{\varphi} + 2\cot\theta\,\dot{\theta}\dot{\varphi} = 0$$

If we multiply the equation on the left across by the appropriate constants, the first two terms are just the rate of change of momentum or angular momentum, and the third term is the force or torque due to gravity. This is Newton's second law. Alternatively, the three terms can be integrated in which case the first two become the kinetic energy of the pendulum, and the third term is its potential energy. That form of course leads to $T + V =$ constant.

The second equation (on the right) resulted from setting equal to zero the d/dt of $p_\varphi = mr_0^2 \sin^2\theta\,\dot{\varphi}$. This quantity p_φ is angular momentum about the vertical axis. Therefore, the second equation reflects conservation of this component of angular momentum. See the discussion of *cyclic* coordinates in Chapter 2 for more about this topic.

Special case #1: $\dot{\varphi} = 0$, then $\ddot{\theta} = -g\sin\theta\,/\,r_0$, a simple pendulum which undergoes one-dimensional simple harmonic motion for small deflections.

Special case #2: $\theta =$ constant; then $\dot{\varphi}^2 = g\,/\,(r_0\cos\theta)$, a conical pendulum where the mass follows a circular path in the horizontal plane, with uniform angular velocity $\dot{\varphi}$.

C. *Goldstein Exercise 1–22:*

Apply eq. (1-70) in Goldstein to a single vertical coordinate z:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} + \frac{\partial \mathcal{F}}{\partial \dot{z}} = 0$$

Here $L = T - V = \frac{1}{2}m\dot{z}^2 - mgz$ and $\mathcal{F} = \frac{1}{2}k\dot{z}^2$. Thus, equation of motion is

$$m\ddot{z} - mg + k\dot{z} = 0 \qquad \text{or} \qquad \frac{dv}{dt} = g - \frac{k}{m}v,$$

where we have changed notation to $v = \dot{z}$. We can now integrate to get:

$$\ln(g - \frac{k}{m}v) = -\frac{k}{m}t + C$$

Using our initial condition of falling from rest, we get $C = \ln g$ and so

$$v = \frac{mg}{k}[1 - \exp(-\frac{k}{m}t)]$$

For $t \to \infty$, the object falls at a terminal speed $v_{\max} = mg/k$.

D. *A particle of mass m is attached to a string which passes through a hole in a table (without friction) and is then fastened to a spring with a force constant k. When the particle is at the hole, the spring is unstretched. Using Lagrange's equations, find the equations of motion in polar coordinates and identify any constants of the motion. Consider only cases where the particle moves on the surface of the table. What is the shape of the path of the particle?*

We position our origin at the hole in the table. The Lagrangian is

$$L = T - V = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \tfrac{1}{2}kr^2$$

Lagrange's eq. for $\theta \quad \Rightarrow$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

which tells us that angular momentum about the vertical axis, $mr^2\dot{\theta} = p_\theta =$constant. (This conclusion could be reached from the outset by noting that θ is *cyclic*.) Also, Lagrange's eq. for r leads to the other eq. of motion in terms of r only:

$$m\ddot{r} - mr\dot{\theta}^2 + kr = 0 \qquad \text{or} \qquad m\ddot{r} - \frac{p_\theta^2}{mr^3} + kr = 0$$

It is clear *a priori* that the energy $T + V$ is also constant.

A relatively easy way to identify the shape of the path in this case is to rewrite the Lagrangian in cartesian coordinates:

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}k(x^2 + y^2)\,.$$

Then Lagrange's equations $\Rightarrow\Rightarrow$ $m\ddot{x} + kx = 0$ and $m\ddot{y} + ky = 0$. We recognize these as the equations for simple harmonic motion. Note that the frequency of oscillation, which is determined by $\omega^2 = k/m$, is the same for x and y. See any text on introductory physics for proof that the resultant of two perpendicular simple harmonic oscillators is an ellipse if both oscillate at the same frequency.

In Chapter 3, you can find alternative ways to show that a particle moving under an attractive central force, as supplied by the spring, follows an elliptic path.